

28 there exist ε_j such that the perturbed matrix

$$29 \quad B = A + \text{diag}(\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{S}$$

30 has characteristic polynomial

$$31 \quad q(t) = t^n + b_{n-1}t^{n-1} + \dots + b_1t + b_0.$$

32 *Proof.* We start from the standard version of Jacobi's formula, [Wikipedia \(2018\)](#):

$$33 \quad \det(K + \varepsilon X) = \det K + \text{tr}(\text{adj}(K)X)\varepsilon + O(\varepsilon^2)$$

34 for $n \times n$ matrices K, X . Here $\text{adj}(K)$ is the adjugate (or adjoint) matrix — the transpose
35 of the matrix of cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$ where M_{ij} is the minor obtained by deleting
36 row i and column j from K . This formula follows directly from the standard formula for the
37 determinant as a sum over permutations σ of products of the form $\text{sign}(\sigma)a_{i,\sigma(i)}$.

38 The first step is to perturb the diagonal so that all diagonal entries a_{ii} are distinct. Having
39 pre-prepared A in this manner, we proceed as follows:

40 Let e_{ij} be the elementary matrix with 1 in the (i, j) position and 0 everywhere else. Put
41 $K = A - tI$ and $X = e_{11}$. The only contribution to the trace of $\text{adj}(K)X$ comes from the
42 $(1, 1)$ position, so we want the cofactor C_{11} for $A - tI$. Since $(-1)^{1+1} = 1$, this is

$$43 \quad \det[A^{[1]} - tI]$$

44 where $A^{[1]}$ is A with row and column 1 deleted. This is the characteristic polynomial of $A^{[1]}$.

45 Let $p^{[i]}$ be the characteristic polynomial of $A^{[i]}$, which is A with row and column i deleted.
46 The same calculation (think of the Taylor expansion or just take $X = \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$) yields

$$47 \quad q(t) = \det(A - tI + \text{diag}(\varepsilon_1, \dots, \varepsilon_n)) = p(t) + \sum_i \varepsilon_i p^{[i]}(t) + O(2)$$

48 where $O(2)$ is of order 2 in the ε_i .

49 By the Implicit Function Theorem it is enough to prove the theorem neglecting the $O(2)$
50 terms. So we have to prove that generically (that is, after a small enough perturbation) the
51 polynomials $p^{[1]}, \dots, p^{[n]}$ are linearly independent. (The ε_i are independent, and there are n
52 of them, the same as the number of coefficients in each $p^{[i]}$, including the leading term $1 \cdot t^{m-1}$.)

53 Expanding and collecting coefficients of powers of t , each coefficient of each $p^{[i]}$ is a poly-
54 nomial in the entries a_{ij} of A . The condition for linear independence is that the determinant
55 Δ of these coefficients (including the leading term with coefficient 1) should not vanish.

56 We claim that Δ defines a codimension-1 subvariety. This follows provided Δ does not
57 vanish identically on the shape space \mathcal{S} . Suppose for a contradiction that it does vanish. Then
58 it vanishes on the diagonal matrices, since these are contained in \mathcal{S} , so some nontrivial linear
59 combination vanishes:

$$60 \quad (\text{SM1.1}) \quad \sum_i \mu_i p^{[i]}(t) \equiv 0.$$

61 However, when A is diagonal,

$$62 \quad p^{[i]}(t) = \prod_{j \neq i} (t - a_{jj}).$$

63 We initially perturbed the diagonal of A (say by $\varepsilon/2$) so that all diagonal elements a_{ii} are
 64 distinct. Now for each i we can substitute $t = a_{ii}$ in (SM1.1). All terms vanish except possibly

$$65 \quad \mu_i p^{[i]}(a_{ii}) = \prod_{j \neq i} (a_{ii} - a_{jj}),$$

66 so $\mu_i = 0$ for all i , contradiction.

67 Therefore the $p^{[i]}$ are linearly independent off the codimension-1 variety $\Delta = 0$, so small
 68 enough ε_j give characteristic polynomials filling an entire neighbourhood of $p(t)$.

69 **Corollary SM1.2.** *All sets of eigenvalues sufficiently close to those of A can be obtained by*
 70 *a diagonal perturbation $A + \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$.*

71 *Proof.* Take $q(t) = (t - \lambda_1) \cdots (t - \lambda_n)$ to be the perturbed polynomial in Theorem SM1.1,
 72 where the λ_j are the required perturbed eigenvalues.

73 SM1.2. Proofs for Codimension One Steady-State Bifurcation.

74 *Proof of Theorem 3.1.:* Fix a fully inhomogeneous network and consider an admissible
 75 system

$$76 \quad \dot{y} = F(y, \lambda)$$

77 for the network, where $y \in \mathbb{R}^n, \lambda \in \mathbb{R}$. Assume $F(0, 0) = 0$ so that $y = 0$ is an equilibrium
 78 when $\lambda = 0$. Because the network is fully inhomogeneous, Corollary SM1.2 implies that at a
 79 codimension-1 steady-state bifurcation the Jacobian $J = (D_y F)_{(0,0)}$ generically has a simple
 80 zero eigenvalue. We call the corresponding eigenvector $v \neq 0$, so $Jv = 0$.

81 Let $v^* \neq 0$ be a null vector for the adjoint J^* . The range of J is the orthogonal complement
 82 of v^* since $\langle v^*, Jw \rangle = \langle J^* v^*, w \rangle = 0$ for any $w \in \mathbb{R}^n$, and the range of J is $(n-1)$ -dimensional.
 83 We claim that

$$84 \quad (\text{SM1.2}) \quad \langle v^*, v \rangle \neq 0.$$

85 To prove the claim assume, for a contradiction, that $\langle v^*, v \rangle = 0$. This implies $v \in \text{range}(J)$,
 86 so there exists $u \neq 0$ such that $Ju = v$ and $J^2 u = Jv = 0$. Since u and v are linearly
 87 independent, zero is not a simple eigenvalue of J , a contradiction.

88 Since $\text{rank}(J) = n-1$, Liapunov-Schmidt reduction shows that the zeros of $F(y, \lambda)$ near the
 89 bifurcation are in one-to-one correspondence with zeros of a single equation $g(x, \lambda)$, where $x \in$
 90 \mathbb{R} . The Liapunov-Schmidt procedure implies that $g_x(0, 0) = 0$. Moreover, $g_{xx}(0, 0)g_\lambda(0, 0) \neq 0$
 91 if and only if the resulting bifurcation is a saddle-node bifurcation.

92 The formulas for computing $g_{xx}(0, 0)$ and $g_\lambda(0, 0)$ are standard (Golubitsky and Schaeffer,
 93 1985, p. 33). In particular,

$$94 \quad \begin{aligned} g_{xx}(0, 0) &= \langle v^*, D^2 F(v, v) \rangle \\ g_\lambda(0, 0) &= \langle v^*, F_\lambda \rangle \end{aligned}$$

95 In coordinates write $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$. Then the k^{th} component of D^2F
 96 is

$$97 \quad [D^2F(v, w)]_k = \sum_{i,j=1}^n \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0) v_i w_j.$$

98 In order to show that saddle-node bifurcations are generic, we must consider the two cases
 99 when either $g_{xx}(0, 0) = 0$ or $g_\lambda(0, 0) = 0$. In each case we must show that there is a generic
 100 perturbation that forces these derivatives to be nonzero.

101 The linear case is straightforward. Suppose that the j^{th} coordinate of v_j^* is nonzero.
 102 Perturb the j^{th} (node) coordinate of F to $G_j(y, \lambda) = F_j(y, \lambda) + \varepsilon \lambda$ and assume $G_k = F_k$ for
 103 $k \neq j$. Note that $F = G = 0$ and $DG = DF$ at $y = \lambda = 0$. It follows that $g_\lambda = \varepsilon v_j^* \neq 0$ when
 104 $\varepsilon \neq 0$.

105 Next, we must show that a generic homogeneous quadratic perturbation of F leads to a
 106 new vector field $G = F + \varepsilon \Phi$ where

$$107 \quad g_{xx}^\varepsilon(0, 0) = \langle v^*, D^2G(v, v) \rangle = \varepsilon \langle v^*, D^2\Phi(v, v) \rangle$$

108 is nonzero. The Jacobian J and therefore v and v^* remain unchanged by this perturba-
 109 tion because we assume Φ to be homogeneous quadratic. Indeed, because the map $\Phi \mapsto$
 110 $\langle v^*, D^2\Phi(v, v) \rangle$ is linear, it is enough to show that

$$111 \quad (\text{SM1.3}) \quad \langle v^*, D^2\Phi(v, v) \rangle \neq 0$$

112 for some admissible Φ , in order to satisfy (SM1.3) for almost all admissible Φ .

113 The quadratic $\Phi = (\phi_1, \dots, \phi_n)$ is admissible if

$$114 \quad \frac{\partial \phi_j}{\partial x_i} = 0$$

115 whenever node j is not connected to node i and $j \neq i$. In particular, a quadratic Φ of the
 116 form $\partial \phi_j / \partial x_i = 0$ for $i \neq j$ is admissible for any network.

117 By (SM1.2) there is a component k such that both $v_k^* \neq 0$ and $v_k \neq 0$. We can therefore
 118 choose $\phi_k = x_k^2/2$ and $\phi_j = 0$ for $j \neq k$ so that for this Φ we have

$$119 \quad g_{xx}^\varepsilon(0, 0) = \varepsilon v_k^* v_k^2 \neq 0$$

120 The bifurcation of the perturbed vector field is therefore a saddle-node.

121 **Lemma SM1.3.** *Fix a fully inhomogeneous network with shape space \mathcal{S} . Let $\hat{\mathcal{S}}$ be the set*
 122 *of matrices $J \in \mathcal{S}$ that have a simple zero eigenvalue. Define $\hat{\mathcal{S}}_i \subseteq \hat{\mathcal{S}}$ to be the set of matrices*
 123 *$J \in \hat{\mathcal{S}}$ with null vector $v = (v_1, \dots, v_n)$ such that $v_i \neq 0$. Let*

$$124 \quad \hat{\mathcal{T}}_{ij} = \{J \in \hat{\mathcal{S}}_i : v_j \neq 0\}.$$

125 *Then for any node j downstream of node i , $\hat{\mathcal{T}}_{ij}$ is open and dense in $\hat{\mathcal{S}}_i$.*

126 *Proof.* The set of $J \in \hat{\mathcal{S}}_i$ that lead to a nonzero v_j is open by the continuous movement of
 127 the null vector v . So it is enough to show that this subset is dense in $\hat{\mathcal{S}}_i$ for all j downstream
 128 of i .

129 Fix $J \in \hat{\mathcal{S}}_i$. If $v_j \neq 0$, the proof is complete, so we can suppose $v_j = 0$. Because node j is
 130 downstream of node i , there exists a path of length m such that $k_0 \rightarrow k_1 \rightarrow \dots \rightarrow k_m$ where
 131 $k_0 = i$, $k_m = j$, and arrows indicate connections between the corresponding nodes.

132 Given that component $v_{k_0} = v_i$ is nonzero, the proof proceeds by constructing a series of
 133 m perturbations that sequentially makes each k_ℓ component of the null vector v nonzero, to
 134 achieve the desired result $v_j \neq 0$.

135 Suppose we have found perturbations 1 through ℓ so that $v_{k_0}, \dots, v_{k_\ell}$ are all nonzero. We
 136 show how to make an arbitrarily small perturbation of J , denoted by \tilde{J} , that makes $v_{k_{\ell+1}} \neq 0$.
 137 We choose the perturbation small enough so that the perturbed Jacobian still lies in $\hat{\mathcal{S}}_i$, and
 138 the nonzero components of v remain nonzero. For convenience, relabel nodes so that $k_{\ell+1}$ is 1
 139 and k_ℓ is 2. Now $v_1 = 0$ and node 1 receives input from node 2 with $v_2 \neq 0$. Each perturbation
 140 is constructed in two stages, as we now describe.

141 *First perturbation:* Given the above labeling, let

$$142 \quad J = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

143 where A is a scalar and D is an $(n-1) \times (n-1)$ matrix. By assumption $v = (0 \ z)^T$ for
 144 $z \in \mathbb{R}^{n-1}$. The condition that v is a null vector becomes

$$145 \quad (\text{SM1.4}) \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} = J \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

146 Now (SM1.4) implies $Bz = 0$ and $Dz = 0$, so D is singular with null vector z .

147 We claim there exists a perturbation of J so that the zero eigenvalue of D is simple. By
 148 Theorem SM1.1 we can choose an admissible $n \times n$ perturbation matrix

$$149 \quad \Psi_\Delta = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix},$$

150 where Δ is diagonal, so that the perturbed matrix $J + \Psi_\Delta$ has a simple eigenvalue λ close to
 151 zero. Then the perturbed matrix

$$152 \quad \tilde{J} = J + \Psi_\Delta - \lambda I = \begin{bmatrix} \tilde{A} & B \\ C & \tilde{D} \end{bmatrix},$$

153 where $\tilde{A} = A - \lambda$ and $\tilde{D} = D + \Delta - \lambda I$, has a simple zero eigenvalue. Moreover, we can choose
 154 the perturbation Ψ_Δ small enough so that the nonzero components of v remain nonzero and
 155 $\tilde{J} \in \hat{\mathcal{S}}_i$.

156 If the new null vector of \tilde{J} has $v_1 \neq 0$, we are done. So we may assume that the null vector
 157 still has the form $\tilde{v} = (0 \ \tilde{z})^T$, which implies that the simple zero eigenvalue is associated with
 158 \tilde{D} . The claim is verified by dropping the tildes on \tilde{A} , \tilde{D} , \tilde{J} , \tilde{v} and \tilde{z} .

159 *Second perturbation:*. Let

$$160 \quad \Phi_E = \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix},$$

161 where $E = (\varepsilon, 0, \dots, 0)$. Since node 2 connects to node 1, Φ_E is a small admissible matrix.
 162 Moreover, $Ez \neq 0$. Consider the small perturbation $\tilde{J} = J + \Phi_E - \rho I$ of J , where ρ is the
 163 simple eigenvalue of the matrix $J + \Phi_E$ near zero. Thus \tilde{J} has a simple zero eigenvalue with
 164 null vector $\tilde{v} = [\tilde{y} \ \tilde{z}]^T$ and

$$165 \quad (\text{SM1.5}) \quad \tilde{J} \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} A - \rho I & B + E \\ C & D - \rho I \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

166 We claim that $\tilde{y} \neq 0$. We argue by contradiction; suppose $\tilde{y} = 0$. Then (SM1.5) reduces to

$$167 \quad (\text{SM1.6}) \quad \begin{aligned} B\tilde{z} + E\tilde{z} &= 0 \\ D\tilde{z} &= \rho\tilde{z} \end{aligned}$$

168 Since ρ is near 0 and D has a simple eigenvalue at 0 with all other eigenvalues bounded away
 169 from 0, it follows that $\rho = 0$ and we can take $\tilde{z} = z$. From (SM1.4) we know that $Bz = 0$;
 170 hence (SM1.6) implies $Ez = 0$. This is a contradiction, so $\tilde{y} \neq 0$.

171 The proof of Lemma SM1.3 does not require the eigenvalues or the eigenvectors of J to
 172 be real. Hence Lemma SM1.3 also holds for matrices in \mathcal{S} that have a pair of simple purely
 173 imaginary eigenvalues. This adaptation, stated without proof in the next lemma, is needed in
 174 Section 3.2.

175 **Lemma SM1.4.** *Fix a fully inhomogeneous network with shape space \mathcal{S} . Let $\hat{\mathcal{S}}$ be the set of*
 176 *matrices $J \in \mathcal{S}$ that have a pair of simple purely imaginary eigenvalues. Define $\hat{\mathcal{S}}_i \subseteq \hat{\mathcal{S}}$ to be*
 177 *the set of matrices $J \in \hat{\mathcal{S}}$ with critical eigenvector $v = (v_1, \dots, v_n)$ such that $v_i \neq 0$. Let*

$$178 \quad \hat{\mathcal{T}}_{ij} = \{J \in \hat{\mathcal{S}}_i : v_j \neq 0\}.$$

179 *Then for any node j downstream of node i , $\hat{\mathcal{T}}_{ij}$ is open and dense in $\hat{\mathcal{S}}_i$.*

180 **Lemma SM1.5.** *Let C be the critical path component associated with the saddle-node bifur-*
 181 *cation. Let v be the associated critical eigenvector. Then the coordinates of v on nodes that*
 182 *are not downstream from C are zero. Generically, the coordinates of v on all nodes that are*
 183 *downstream from C (including C) are nonzero.*

184 *Proof.* Because the zero eigenvalue at the saddle-node bifurcation is simple, it is associated
 185 with a unique path component C . By Theorem 2.9, the corresponding zero eigenvector v has
 186 zero components on nodes that are not downstream from C . Since v has at least one nonzero
 187 component on C , Lemma SM1.3 implies that v has nonzero components on C and on all nodes
 188 downstream.

189 *Proof of Theorem 3.3:* The growth rates follow from Lemma SM1.5 and Remark 3.2.

190 **SM1.3. Proofs for Codimension One Hopf Bifurcation.** We first consider the special
 191 case of a directed ring, and then parlay this case into a proof of the general result.

192 **Lemma SM1.6.** *Nondegenerate Hopf bifurcation can occur for suitable admissible vector*
 193 *fields in a directed ring with more than one node.*

194 *Proof.* Consider a directed ring of nodes $1, \dots, m$ with $1 \rightarrow 2, \dots, m \rightarrow 1$. Admissible
 195 vector fields for this ring have the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_m) \\ \dot{x}_2 &= f_2(x_2, x_1) \\ &\vdots \\ \dot{x}_m &= f_m(x_m, x_{m-1}) \end{aligned} \tag{SM1.7}$$

197 Assume that (SM1.7) has an equilibrium at the origin; that is, $f_j(0) = 0$ for all j . We claim
 198 that the $m \times m$ Jacobian of (SM1.7) at the origin can be chosen to have a pair of simple
 199 complex conjugate purely imaginary eigenvalues, and no other imaginary eigenvalues.

200 In block form let L be the $m \times m$ matrix

$$L = \begin{bmatrix} 0 & 1 \\ I_{m-1} & 0 \end{bmatrix}.$$

202 The characteristic polynomial of L is $p(\lambda) = \det(\lambda I_m - L) = \lambda^m - 1$ and the eigenvalues of L
 203 are the m^{th} roots of unity. For each m the matrix L has simple complex conjugate eigenvalues,
 204 so there exists μ such that $J = L - \mu I_m$ has simple purely imaginary eigenvalues and no other
 205 imaginary eigenvalues. The standard Hopf bifurcation theorem implies that adding λI_m to
 206 the vector field leads to a nondegenerate Hopf bifurcation; that is, to the desired $\sqrt{\lambda}$ growth
 207 rate of small amplitude periodic solutions.

208 **Lemma SM1.7.** *Hopf bifurcation in a path component H , at a pair of simple complex con-*
 209 *jugate purely imaginary eigenvalues, is possible for some admissible map if and only if the*
 210 *number of nodes in that component satisfies $n_H > 1$.*

211 *Proof.* For $n_H = 1$, Hopf bifurcation is not possible. We therefore show that for $n_H > 1$,
 212 there exists an admissible Jacobian J with one pair of simple purely imaginary eigenvalues
 213 and all other eigenvalues off of the imaginary axis.

214 Fix a path component H with $n_H > 1$ nodes. We construct a directed ring within that
 215 path component as follows. Given any two distinct nodes ℓ and k in H , there exists a directed
 216 loop $\ell \rightarrow \dots \rightarrow k$ and $k \rightarrow \dots \rightarrow \ell$. Consider a loop of minimal length m . If any node occurs
 217 twice (except where the ends join) the segment in between is a smaller loop. So a minimal
 218 loop consists of distinct nodes, forming a closed ring. In particular, there are no connections
 219 between distinct nodes in the ring, except for the unidirectional nearest neighbor ones.

220 Order the nodes in the ring by $1, \dots, m$. Consider admissible vector fields such that the
 221 coordinate function $f_j \equiv 0$ when $j > m$ and f_j has the form in (SM1.7) for $1 \leq j \leq m$. By
 222 Lemma SM1.6 these admissible vector fields can have an equilibrium at which the Jacobian
 223 J has simple purely imaginary eigenvalues. However, 0 occurs $n - m$ times as an eigenvalue
 224 of J .

225 Theorem SM1.1 implies that we can perturb the diagonal entries of J to make the 0
 226 eigenvalues nonzero while fixing the purely imaginary pair of eigenvalues. The Jacobian J
 227 constructed in this way is admissible, and it has exactly one pair of simple purely imaginary
 228 eigenvalues and no other imaginary eigenvalues.

229 *Proof of Theorem 3.6:* Fix a fully inhomogeneous network and consider the network ad-
 230 missible system

$$231 \quad \dot{y} = F(y, \lambda)$$

232 for $y \in \mathbb{R}^n, \lambda \in \mathbb{R}$. Assume $F(0, \lambda) = 0$ so that $y = 0$ is a steady-state solution for all λ .
 233 By Lemma SM1.7, purely imaginary eigenvalues associated with each path component H are
 234 possible as long as $n_H > 1$. Moreover, we can assume these eigenvalues are simple and all
 235 other eigenvalues are off the imaginary axis. Without loss of generality we can assume at
 236 codimension-1 Hopf bifurcation that $J = (D_y F)_{(0,0)}$ has simple eigenvalues $\pm i$ and no other
 237 imaginary eigenvalues.

238 Define the eigenvectors c and d by $Jc = -ic$ and $J^T d = id$, where the superscript T
 239 denotes transpose. Using the inner product

$$240 \quad (\text{SM1.8}) \quad \langle w, v \rangle = \bar{w}^T v,$$

241 where the overbar denotes complex conjugate, we can choose d such that $\langle d, c \rangle = 2$ (Golubitsky
 242 and Schaeffer, 1985, p. 346). In particular,

$$243 \quad \langle d, c \rangle \neq 0.$$

244 Since the critical eigenvalues of J are simple, Liapunov-Schmidt reduction shows that near
 245 bifurcation the small amplitude periodic orbits of $\dot{y} = F(y, \lambda)$ are in one-to-one correspondence
 246 with zeros of a single equation $g(x, \lambda) = r(x^2, \lambda)x = 0$, where $x \in \mathbb{R}$. By the Liapunov-
 247 Schmidt procedure, $r_z(0, 0)r_\lambda(0, 0) \neq 0$ (where $z = x^2$) if and only if the resulting bifurcation
 248 is a nondegenerate Hopf bifurcation. The formulas for computing $r_z(0, 0)$ and $r_\lambda(0, 0)$ are
 249 standard (Golubitsky and Schaeffer, 1985, p. 352), and we assume that $r_\lambda(0, 0) \neq 0$.

250 To show that Hopf bifurcations are nondegenerate, we consider the case $r_z(0, 0) = 0$,
 251 and prove that a generic homogeneous cubic perturbation of F leads to a new vector field
 252 $G = F + \varepsilon\Psi$ such that the new cubic coefficient in the reduction $r_z^\varepsilon(0, 0) \neq 0$. In this case the
 253 coefficient can be computed as

$$254 \quad r_z^\varepsilon(0, 0) = \frac{1}{16} \text{Re} \langle d, (D^3 G)(c, c, \bar{c}) \rangle = \frac{\varepsilon}{16} \text{Re} \langle d, (D^3 \Psi)(c, c, \bar{c}) \rangle$$

255 The Jacobian J and therefore c and d remain unchanged by the perturbation because we
 256 assume Ψ to be homogeneous cubic.

257 Because $\text{Re} \langle d, c \rangle = 2$, there must be some node k such that $\text{Re}(\bar{d}_k c_k) \neq 0$ and $|c_k| \neq 0$.
 258 Choose $\Psi_k = \frac{1}{6}x_k^3$ and $\Psi_j = 0$ for $j \neq k$. Then

$$259 \quad r_z^\varepsilon(0, 0) = \frac{\varepsilon}{16} \text{Re}(\bar{d}_k c_k c_k \bar{c}_k) = \frac{\varepsilon}{16} |c_k|^2 \text{Re}(\bar{d}_k c_k) \neq 0$$

260 as desired. This perturbation is admissible, since for every node j the variable x_j appears in
 261 f_j in (1.1).

262 **Lemma SM1.8.** *Let H be the critical path component associated with a nondegenerate Hopf*
 263 *bifurcation, and let v be the associated critical eigenvector. Then the coordinates of v on nodes*
 264 *that are not downstream from H are zero. Generically, the coordinates of v on all nodes that*
 265 *are downstream from H are nonzero.*

266 *Proof.* Since the purely imaginary eigenvalues are simple at the Hopf bifurcation, we can
 267 associate it with a unique path component H . The fact that the coordinates of v that are
 268 not downstream from H are zero follows from Theorem 2.9. By Lemma SM1.4, generically
 269 all components of v that are downstream from H are nonzero.

270 *Proof of Theorem 3.7.:* This follows immediately from Lemma SM1.8.

271 **SM2. Overview of Singularity Theory.** The analysis of codimension-2 mode interactions
 272 in the next four sections relies on techniques from singularity theory. We summarize the
 273 main concepts and results here. We follow the approach to bifurcation problems in Golubitsky
 274 and Schaeffer (1985); Golubitsky, Stewart and Schaeffer (1988), but we do so without a
 275 distinguished parameter. These sources should be consulted for further details and proofs.

276 The analysis of the four codimension-2 mode interactions (steady-state/steady-state, steady-
 277 state/Hopf, Hopf/steady-state, and Hopf/Hopf) reduce to functions $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that satisfy
 278 a feedforward structure $F(x, y) = (f(x), g(x, y))$. In addition, F commutes with the action of
 279 a symmetry group on \mathbb{R}^2 in the three interactions involving Hopf modes.

280 Singularity theory is about the local topological structure of classes of C^∞ smooth maps

$$281 \quad F : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

282 near some point. By translation, we take this point to be $0 \in \mathbb{R}^m$ and assume $F(0) = 0$.
 283 The local structure is captured by introducing the following notion. Two such maps F, G are
 284 *germ-equivalent* if their restrictions to some open neighborhood $U \subseteq \mathbb{R}^m$ of 0 are equal; that
 285 is, $F(X) = G(X)$ for all $X \in U$. A *germ* is a germ-equivalence class. We define a germ by
 286 specifying a representative map, and identify the germ with this map, bearing in mind that
 287 only local information near 0 is meaningful. In particular, derivatives $D^k F|_{X=0}$ of F at 0 are
 288 meaningful concepts for the germ of F , and so is the Taylor series of F near 0.

289 With this understood, we can henceforth omit ‘germ’ and refer to maps and functions.
 290 All of these are assumed smooth, and we mainly require the case $m = n = 2$.

291 Singularity theory uses changes of coordinates to simplify the form of F , where possible.
 292 These changes of coordinates preserve the number of solutions (zeros of F), and the type of
 293 solutions (if symmetry is present). To do this, define two problems $F, G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to be
 294 *contact equivalent* if

$$295 \quad G(X) = S(X)F(\Phi(X))$$

296 where the smooth map $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism and the smooth map

$$297 \quad S : \mathbb{R}^2 \rightarrow \mathbb{GL}(2)$$

298 where $\mathbb{GL}(2)$ is the group of invertible real 2×2 matrices. The equivalence is *strong* if $\Phi(0) = 0$.

299 Contact equivalence preserves the topology of the zero set of F . It is the most general
 300 form of equivalence with this property, and it has technical advantages over any stronger form
 301 of equivalence.

302 The methods of singularity theory usually have to be adapted to any specific context,
 303 imposing extra conditions to ensure that the equivalences preserve any relevant structure. As
 304 we see in the next four sections, contact equivalence must be suitably modified in each of the
 305 four mode interactions to preserve the feedforward structure and the relevant symmetry.

306 *Normal form theory.* The first main objective is to use an suitable equivalence to transform
 307 a given map F into a simple polynomial map, a *normal form*. This is not always possible,
 308 but it can be done for ‘almost all’ maps, namely, those of finite codimension, see (SM2.1).
 309 To achieve this we consider ‘infinitesimal’ perturbations. Consider a one-parameter family of
 310 strong equivalences

$$311 \quad G(X, \varepsilon) = S(X, \varepsilon)F(\Phi(X, \varepsilon))$$

312 where $\varepsilon \in \mathbb{R}$ is small. Differentiate with respect to ε (shown by a dot) and evaluate at $\varepsilon = 0$.
 313 We get

$$314 \quad \dot{G}(x, 0) = \dot{S}(X, 0)F(X) + (DF)_X \dot{\Phi}(X, 0)$$

315 We therefore define the *restricted tangent space* of F to consist of all possible \dot{G} ; that is,

$$316 \quad RT(F) = \{SF + (DF)\Phi\}$$

317 where $S(X)$ is an arbitrary 2×2 matrix for each X and $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an arbitrary map
 318 that satisfies $\Phi(0) = 0$. We now have:

319 **Theorem SM2.1 (Tangent Space Constant Theorem).** *Let F be a vector field. Suppose*
 320 *there exists $p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that*

$$321 \quad RT(F + \varepsilon p) = RT(F)$$

322 *for all $\varepsilon \in [0, 1]$. Then $F + \varepsilon p$ is strongly equivalent to F for all $\varepsilon \in [0, 1]$.*

323 See Golubitsky and Schaeffer (1985) Chapter II Theorem 2.2 when $n = 1$, and Golubitsky,
 324 Stewart and Schaeffer (1988) Chapter XIV Theorem 3.1 for the general case. We can ap-
 325 ply Theorem SM2.1 to construct normal forms, and to solve the *recognition problem*: using
 326 conditions on Taylor coefficients to characterize when F has the normal form concerned.

327 The proof of Theorem SM2.1 can be adapted to prove analogous theorems for each of
 328 the mode interactions. Alternatively, the appropriate tangent space constant theorem for
 329 each mode interaction follows from general results of Damon (1988). The principal difficulty
 330 in applying Theorem SM2.1 is the computation of $RT(F)$. This computation is simplified
 331 by using its algebraic structure (a module over a system of rings) and Nakayama’s Lemma
 332 (Golubitsky and Guillemin (1973); Gibson (1979); Martinet (1982)), which we briefly recall:

333 **Lemma SM2.2 (Nakayama’s Lemma).** *Let \mathcal{R} be a commutative ring with unit, with an*
 334 *ideal \mathcal{I} such that whenever $r \in \mathcal{I}$ the element $1 + r$ is invertible in \mathcal{R} . Let \mathcal{M} be a finitely*
 335 *generated \mathcal{R} -module, with a submodule \mathcal{N} . Then the condition*

$$336 \quad \mathcal{N} + \mathcal{I}\mathcal{M} = \mathcal{M}$$

337 *implies that $\mathcal{N} = \mathcal{M}$.*

338 The rings that we use are rings \mathcal{E} of germs of \mathcal{C}^∞ functions at the origin and the associated
 339 ideals \mathcal{M} of functions in \mathcal{E} that vanish at the origin. The modules over these rings are modules
 340 of vector mappings denoted by $\vec{\mathcal{E}}$. The exact definition of these spaces (that is, the coordinates
 341 X on which the functions are defined) are context dependent. See (Golubitsky and Guillemin,
 342 1973, p. 103) or (Golubitsky and Schaeffer, 1985, Chapter 2).

343 The next four sections carry out these calculations under the assumption that the corre-
 344 sponding tangent space constant theorem is valid.

345 *Unfolding theory.* The other main topic we need is unfolding theory, which determines all
 346 possible perturbations of F (of finite codimension) in a sense we now explain. A k -parameter
 347 *unfolding* of F is a smooth map

$$348 \quad \tilde{F} : \mathbb{R}^2 \times \mathbb{R}^k \rightarrow \mathbb{R}^2$$

349 such that

$$350 \quad \tilde{F}(X, 0) = F(0)$$

351 Let $\tilde{H}(X, \beta)$ be an l -parameter unfolding of F . Then \tilde{H} *factors through* \tilde{F} if

$$352 \quad \tilde{H}(X, \beta) = S(X, \beta)\tilde{F}(\Phi(X, \beta), A(\beta))$$

353 where $A : \mathbb{R}^l \rightarrow \mathbb{R}^k$, $A(0) = 0$, and $S(X, 0) = I$, $\Phi(X, 0) = X$. An unfolding is *versal* if every
 354 unfolding factors through it. It is *universal* if it is versal and the number of parameters is
 355 minimal among all versal unfoldings. This minimal number is the *codimension*

$$356 \quad (\text{SM2.1}) \quad \text{codim}(F).$$

357 The *tangent space* of F is

$$358 \quad T(F) = \{SF + (DF)\Phi\},$$

359 where the diffeomorphism Φ is an equivalence, but not necessarily a strong equivalence. That
 360 is, $\Phi(0)$ need not equal 0.

361 The codimension of F is equal to the codimension of the *tangent space* $T(F)$, which
 362 contains $RT(F)$ but may be larger. We refer to [Golubitsky, Stewart and Schaeffer \(1988\)](#)
 363 Chapter XV Section 2 for a discussion and definition.

364 Finally, we state a criterion for a universal unfolding to exist:

365 **Theorem SM2.3.** *An unfolding \tilde{F} of F is a universal unfolding of F if and only if*

$$366 \quad \vec{\mathcal{E}}_X = T(F) \oplus \mathbb{R}\{\tilde{F}_{\alpha_1}(X, 0), \dots, \tilde{F}_{\alpha_k}(X, 0)\}.$$

367 **Corollary SM2.4.** *The codimension of F is equal to the codimension of $T(F)$ in $\vec{\mathcal{E}}_X$.*

368 The proofs of the theorems for the four mode interaction cases that are analogous to
 369 Theorem [SM2.3](#) follow from [Damon \(1988\)](#) and these corresponding theorems are used in the
 370 next four sections. Generally speaking, a singularity theory analysis proceeds by computing
 371 $RT(F)$, determining a normal form \hat{F} of F , computing $T(F)$ based on the computation of
 372 $RT(\hat{F})$, and finally determining a universal unfolding of \hat{F} .

373 **SM3. Steady-State/Steady-State Mode Interaction: Proofs.** This section outlines proofs
 374 based on singularity theory, as reviewed in Section [SM2](#), of the main results in Section [5.1](#) on
 375 steady-state/steady-state mode interaction. We first compute the restricted tangent space for
 376 the center manifold dynamics [\(5.1\)](#) associated with this mode interaction, and use this result
 377 to prove Theorem [5.3](#): that [\(5.1\)](#) can be transformed to the normal form [\(5.5\)](#). We then use a
 378 complement of the unrestricted tangent space of [\(5.5\)](#) to identify the universal unfolding [\(5.7\)](#),
 379 proving Theorem [5.6](#).

380 **SM3.1. Restricted Tangent Space for SS/SS Mode Interaction.** The restricted tangent
 381 space of a map F , denoted $RT(F)$, is obtained from $\frac{d}{d\tau}\Gamma_\tau(F)|_{\tau=0}$, where Γ_τ is a one-parameter
 382 family of strong equivalences (as in Definition 5.2) with $\Gamma_0(F)(x, y) = F(x, y)$.

383 For technical reasons we use a version of singularity theory adapted to maps of the form
 384 $(f(x), g(x, y))$. These maps are analyzed using a special case of the general concept of a *system*
 385 *of rings* and an associated *system of modules*, as defined in Damon (1984) p. 242–243. In this
 386 case the key step is to work with a pair of rings $(\mathcal{E}_x, \mathcal{E}_{x,y})$ instead of a single ring. In place of
 387 a module over a ring, we use a direct sum $M_1 \oplus M_2$ where M_1 is a module over \mathcal{E}_x and M_2 is
 388 a module over $\mathcal{E}_{x,y}$. Tangent spaces and restricted tangent spaces are defined by analogy with
 389 the case of a single ring and module.

390 In Lemma SM3.1 we show that $RT(F)$ is a system of modules over the system of rings
 391 $(\mathcal{E}_x, \mathcal{E}_{x,y})$. The tangent space constant theorem that is analogous to Theorem SM2.1 states
 392 that if

$$393 \quad RT(F + \tau p) = RT(F)$$

394 for all $\tau > 0$, then $F + \tau p$ is strongly equivalent to F . In the context of systems of rings, this
 395 theorem follows from Damon (1984, 1988). See also Dangelmayr and Stewart (1985).

396 **Lemma SM3.1.** *Let $F = (f(x), g(x, y))$ be a map in $(\mathcal{E}_x, \mathcal{E}_{x,y})$. A map $G \in (\mathcal{E}_x, \mathcal{E}_{x,y})$ is in*
 397 *$RT(F)$ if and only if there exist maps $P_i(x) \in \mathcal{E}_x$ and $Q_j(x, y) \in \mathcal{E}_{x,y}$ such that*

$$398 \quad G(x, y) = P_1 \begin{bmatrix} f \\ 0 \end{bmatrix} + P_2 \begin{bmatrix} xf_x \\ xg_x \end{bmatrix} + Q_1 \begin{bmatrix} 0 \\ f \end{bmatrix} + Q_2 \begin{bmatrix} 0 \\ g \end{bmatrix} + Q_3 \begin{bmatrix} 0 \\ xg_y \end{bmatrix} + Q_4 \begin{bmatrix} 0 \\ yg_y \end{bmatrix}$$

399 *Proof.* The general form of strong equivalence is given in (5.2). Define a one-parameter
 400 family of strong equivalences by

$$401 \quad (\text{SM3.1}) \quad \Gamma_\tau(F)(x, y) = \begin{bmatrix} a(x, \tau) & 0 \\ b(x, y, \tau) & c(x, y, \tau) \end{bmatrix} \begin{bmatrix} f(\phi(x, \tau)) \\ g(\phi(x, \tau), \psi(x, y, \tau)) \end{bmatrix},$$

402 where $\Gamma_0(F) = F$ and $\Gamma_\tau(F)(0, 0) = (0, 0)$. Then

$$403 \quad (\text{SM3.2}) \quad \begin{aligned} a(x, 0) = 1 & \quad b(x, y, 0) = 0 & \quad c(x, y, 0) = 1 & \quad \phi(x, 0) = x & \quad \psi(x, y, 0) = y \\ \phi(0, \tau) = 0 & & & & \quad \psi(0, 0, \tau) = 0 \end{aligned}$$

404 We compute the restricted tangent space by differentiating (SM3.1) with respect to τ (indi-
 405 cated by a dot) and evaluating at $\tau = 0$, obtaining

$$406 \quad (\text{SM3.3}) \quad \begin{aligned} \dot{\Gamma}_0(F)(x, y) = & \dot{a}(x, 0) \begin{bmatrix} f(x) \\ 0 \end{bmatrix} + \dot{b}(x, y, 0) \begin{bmatrix} 0 \\ f(x) \end{bmatrix} + \dot{c}(x, y, 0) \begin{bmatrix} 0 \\ g(x, y) \end{bmatrix} \\ & + \dot{\phi}(x, 0) \begin{bmatrix} f_x(x) \\ g_x(x, y) \end{bmatrix} + \dot{\psi}(x, y, 0) \begin{bmatrix} 0 \\ g_y(x, y) \end{bmatrix} \end{aligned}$$

407 We use (SM3.2) to conclude that $\dot{a}(x, 0)$, $\dot{b}(x, y, 0)$ and $\dot{c}(x, y, 0)$ are arbitrary, whereas $\dot{\phi}(x, 0) =$
 408 $x\eta(x)$ and $\dot{\psi}(x, y, 0) = x\sigma(x, y) + y\nu(x, y)$ for arbitrary functions η, σ, ν . The restricted tan-
 409 gent space is therefore

$$410 \quad \left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} xf_x \\ xg_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ f \end{bmatrix}, \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ xg_y \end{bmatrix}, \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \right\rangle_{\{x, y\}}$$

411 Here the notations $\langle \cdots \rangle_{\{x\}}$ and $\langle \cdots \rangle_{\{x,y\}}$ indicate generators of a module over the rings \mathcal{E}_x
 412 and $\mathcal{E}_{x,y}$, respectively.

413 **SM3.2. Normal Form for SS/SS Mode Interaction.** We prove Theorem 5.3 by showing
 414 that F can be transformed to the normal form (5.5). We do this in two steps. First, in
 415 Lemma SM3.2 we explicitly transform the linear and quadratic terms of F into the normal
 416 form (5.5); then we use the tangent space constant theorem to transform away terms of order
 417 three and higher.

418 The defining conditions for a steady-state steady-state mode interaction imply that to
 419 quadratic order F takes the form $F_2 = (f_2, g_2)$, where

$$420 \quad (\text{SM3.4}) \quad \begin{aligned} f_2(x) &= px^2 \\ g_2(x, y) &= qx + rx^2 + sxy + ty^2. \end{aligned}$$

421 **Lemma SM3.2.** Any map of the form (SM3.4) with $p, q, t \neq 0$ is strongly equivalent to the
 422 normal form $\hat{F} = (\hat{f}, \hat{g})$ where

$$423 \quad (\text{SM3.5}) \quad \begin{aligned} \hat{f}(x) &= \varepsilon_p x^2 \\ \hat{g}(x, y) &= \varepsilon_q x + \varepsilon_t y^2, \end{aligned}$$

424 and $\varepsilon_* = \text{sign}(\ast)$.

425 *Proof.* As we are interested only in terms up to second order in x and y , we take the
 426 truncated forms of the transformation functions used to define equivalence in (5.2) to be

$$427 \quad \phi(x) = \alpha x \quad \psi(x, y) = \beta x + \gamma y \quad a(x) = \delta \quad b(x, y) = \sigma \quad c(x, y) = \rho.$$

428 Now

$$429 \quad (\text{SM3.6}) \quad \begin{bmatrix} \hat{f}(x) \\ \hat{g}(x, y) \end{bmatrix} = \begin{bmatrix} \delta p \alpha^2 x^2 \\ \sigma p \alpha^2 x^2 + \rho (q \alpha x + r \alpha^2 x^2 + s \alpha x (\beta x + \gamma y) + t (\beta x + \gamma y)^2) \end{bmatrix}$$

430 Combining terms in (SM3.6), the transformed coefficients are

$$431 \quad \begin{aligned} \hat{p} &= \delta \alpha^2 p \\ \hat{q} &= \rho \alpha q \\ \hat{r} &= \sigma \alpha^2 p + \rho \alpha^2 r + \rho \alpha \beta s + \rho \beta^2 t \\ \hat{s} &= \rho \alpha \gamma s + 2 \rho \beta \gamma t \\ \hat{t} &= \rho \gamma^2 t \end{aligned}$$

432 We assumed that $p \neq 0$, $t \neq 0$, $q \neq 0$ in f and g . Thus we can simplify the system so that
 433 $\hat{p} = \varepsilon_p$, $\hat{q} = \varepsilon_q$, $\hat{t} = \varepsilon_t$ where $\varepsilon_{(\ast)} = \text{sign}(\ast)$. Additionally, we can impose the conditions $\hat{r} = 0$

434 and $\hat{s} = 0$, leading to the transformation

$$\begin{aligned}
435 \quad & \rho = \frac{\varepsilon_t}{\gamma^2 t} \\
436 \quad & \alpha = \frac{\varepsilon_q}{\rho q} = \frac{\varepsilon_q \gamma^2 t}{\varepsilon_t q} \\
437 \quad & \delta = \frac{\varepsilon_p}{\alpha^2 p} = \frac{\varepsilon_t^2 \varepsilon_p q^2}{\varepsilon_p^2 \gamma^4 t^2 p} \\
438 \quad & \beta = -\frac{\alpha s}{2t} = -\frac{\varepsilon_q \gamma^2 s}{2\varepsilon_t q} \\
439 \quad & \sigma = -\frac{\rho}{\alpha^2 p} (\alpha^2 r + \alpha \beta s + \beta^2 t) = -\frac{\varepsilon_t}{\gamma^2 t p} \left(r - \frac{s^2}{4t} \right). \\
440
\end{aligned}$$

441 Here $\gamma > 0$ is a free parameter. We require $\delta, \rho, \alpha, \gamma > 0$ so that the transformation preserves
442 the stabilities of steady states. Were we free to choose the signs of δ, α, ρ , we could have
443 transformed $\hat{p}, \hat{q}, \hat{t}$ to $+1$. Applying the transformation specified above to (SM3.4) produces
444 (SM3.5).

445 *Proof of Theorem 5.3.* By Lemma SM3.2 a general map $F(x, y) = (f(x), g(x, y))$ satisfy-
446 ing the defining and nondegeneracy conditions is strongly equivalent to (5.5) modulo terms
447 of order three or higher. That is, F is equivalent to $\tilde{F} = \hat{F} + \dots$, where \dots indicates terms
448 of order three or higher. Using the tangent space constant theorem we may also remove
449 terms of order three and higher by a suitable transformation. Specifically, we show that
450 $RT(\tilde{F}) = RT(\hat{F})$.

451 First, we claim that the restricted tangent space of the normal form (\hat{f}, \hat{g}) is

$$452 \quad (SM3.7) \quad RT(\hat{F}) = \left[\begin{array}{c} \mathcal{M}_x^2 \\ 0 \end{array} \right] \oplus \left[\begin{array}{c} 0 \\ \mathcal{M}_{x,y}^2 + \mathbb{R}\{x\} \end{array} \right],$$

453 which is a system of modules over the system of rings $(\mathcal{E}_x, \mathcal{E}_{x,y})$. By Lemma SM3.1, the
454 restricted tangent space for the normal form \hat{F} in (5.5) is

$$455 \quad RT(\hat{F}) = \left\langle \left[\begin{array}{c} \varepsilon_p x^2 \\ 0 \end{array} \right], \left[\begin{array}{c} 2\varepsilon_p x^2 \\ \varepsilon_q x \end{array} \right] \right\rangle_{\{x\}} \oplus \left\langle \left[\begin{array}{c} 0 \\ \varepsilon_p x^2 \end{array} \right], \left[\begin{array}{c} 0 \\ \varepsilon_q x + \varepsilon_t y^2 \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t xy \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t y^2 \end{array} \right] \right\rangle_{\{x,y\}} \quad \blacksquare$$

456 By linear combinations of the vectors, we can reduce this to

$$457 \quad RT(\hat{F}) = \left\langle \left[\begin{array}{c} x^2 \\ 0 \end{array} \right] \right\rangle_{\{x\}} \oplus \left\langle \left[\begin{array}{c} 0 \\ x \end{array} \right], \left[\begin{array}{c} 0 \\ x^2 \end{array} \right], \left[\begin{array}{c} 0 \\ y^2 \end{array} \right], \left[\begin{array}{c} 0 \\ xy \end{array} \right] \right\rangle_{\{x,y\}}.$$

458 The restricted tangent space is therefore as in (SM3.7).

459 Next, we consider higher order maps $n \in \mathcal{M}_x^3$ and $m \in \mathcal{M}_{x,y}^3$; that is, $\tilde{F} = \hat{F} + (n, m)$.
460 We use Nakayama's Lemma (Lemma SM2.2) to prove that $RT(\tilde{F}) = RT(\hat{F})$. It follows that
461 $\tilde{F} = (\hat{f} + n, \hat{g} + m)$ is strongly equivalent to the normal form \hat{F} and hence that F is strongly
462 equivalent to \hat{F} , as desired. Specifically, we set

$$463 \quad (SM3.8) \quad \begin{aligned} \tilde{f}(x) &= \varepsilon_p x^2 + n(x) \\ \tilde{g}(x, y) &= \varepsilon_q x + \varepsilon_t y^2 + m(x, y) \end{aligned}$$

464 and compute

$$465 \quad (SM3.9) \quad RT(\tilde{F}) = \left\langle \left[\begin{array}{c} \varepsilon_p x^2 + n \\ 0 \end{array} \right], \left[\begin{array}{c} 2\varepsilon_p x^2 + xn_x \\ \varepsilon_q x + xm_x \end{array} \right] \right\rangle_{\{x\}} \oplus \\ \left\langle \left[\begin{array}{c} 0 \\ \varepsilon_p x^2 + n \end{array} \right], \left[\begin{array}{c} 0 \\ \varepsilon_q x + \varepsilon_t y^2 + m \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t xy + xm_y \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t y^2 + ym_y \end{array} \right] \right\rangle_{\{x,y\}}.$$

466 By (SM3.7) each generator of $RT(\tilde{F})$ in (SM3.9) is in $RT(\hat{F})$. Hence $RT(\tilde{F}) \subseteq RT(\hat{F})$.
 467 Next we apply Nakayama's Lemma to prove $RT(\hat{F}) \subseteq RT(\tilde{F})$, for which we must show
 468 that $RT(\hat{F}) \subseteq RT(\tilde{F}) + (\mathcal{M}_x, \mathcal{M}_{x,y})RT(\hat{F})$. The generators of $RT(\hat{F})$ over the system of
 469 rings are

$$470 \quad \left\langle \left[\begin{array}{c} x^2 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ x \end{array} \right], \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] \right\rangle = \left[\begin{array}{c} \mathcal{M}_x^2 \\ \mathcal{M}_{x,y}^2 + \mathbb{R}\{x\} \end{array} \right]$$

471 and

$$472 \quad (\mathcal{M}_x, \mathcal{M}_{x,y})RT(\hat{F}) = \left[\begin{array}{c} \mathcal{M}_x^3 \\ \mathcal{M}_{x,y}^3 + \mathcal{M}_{x,y}\{x\} \end{array} \right].$$

473 Thus we must show that

$$474 \quad \left\langle \left[\begin{array}{c} x^2 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ x \end{array} \right], \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] \right\rangle \subseteq RT(\tilde{F}) + \left\langle \left[\begin{array}{c} x^3 \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ x^2 \end{array} \right], \left[\begin{array}{c} 0 \\ xy \end{array} \right], \left[\begin{array}{c} 0 \\ y^3 \end{array} \right] \right\rangle,$$

475 which follows from

$$476 \quad \left[\begin{array}{c} x^2 \\ 0 \end{array} \right] = \left[\begin{array}{c} x^2 + n \\ 0 \end{array} \right] - \left[\begin{array}{c} n \\ 0 \end{array} \right] \in RT(\tilde{F}) + \left[\begin{array}{c} \mathcal{M}_x^3 \\ 0 \end{array} \right]$$

477

$$478 \quad \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] = \left[\begin{array}{c} 0 \\ y^2 + ym_y \end{array} \right] - \left[\begin{array}{c} 0 \\ ym_y \end{array} \right] \in RT(\tilde{F}) + \left[\begin{array}{c} 0 \\ \mathcal{M}_{x,y}^3 \oplus \mathcal{M}_{x,y}\langle x \rangle \end{array} \right]$$

479

$$480 \quad \left[\begin{array}{c} 0 \\ x \end{array} \right] = \left[\begin{array}{c} 0 \\ x + y^2 + m \end{array} \right] - \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] - \left[\begin{array}{c} 0 \\ m \end{array} \right] \in RT(\tilde{F}) + \left[\begin{array}{c} 0 \\ \mathcal{M}_{x,y}^3 \oplus \mathcal{M}_{x,y}\langle x \rangle \end{array} \right].$$

481 Therefore the restricted tangent space of the perturbed system is identical to the restricted
 482 tangent space of the original system, so the two are equivalent.

483 The next step is to compute the codimension of the restricted tangent space to ensure
 484 that it is finite. A complement of the restricted tangent space of (5.5) in $(\mathcal{E}_x, \mathcal{E}_{x,y})$ is

$$485 \quad \mathbb{R} \left\{ \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \left[\begin{array}{c} x \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \end{array} \right], \left[\begin{array}{c} 0 \\ y \end{array} \right] \right\}$$

486 so the codimension of $RT(\hat{F})$ is 4.

487 **SM3.3. Tangent Spaces for SS/SS Mode Interaction.** In order to find a universal un-
 488 folding we must compute a complement of the unrestricted tangent space, which is generated
 489 by relaxing the constraint that the origin remains fixed under coordinate transformations.

490 **Lemma SM3.3.** *The tangent space of a map $F = (f, g)$ where $f \in \mathcal{E}_x$ and $g \in \mathcal{E}_{x,y}$ is*

$$491 \text{ (SM3.10)} \quad T(F) = RT(F) \oplus \mathbb{R} \left\{ \begin{bmatrix} f_x(x) \\ g_x(x, y) \end{bmatrix}, \begin{bmatrix} 0 \\ g_y(x, y) \end{bmatrix} \right\}.$$

492 *Proof.* Computing the tangent space is similar to computing the restricted tangent space
 493 as in Lemma SM3.1, except that now we do not require the origin to be fixed by the coordinate
 494 transformation. That is, $\dot{\phi}(x, 0)$ and $\dot{\psi}(x, y, 0)$ in (SM3.3) can be arbitrary functions. Hence
 495 the tangent space is

$$496 \quad \left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} f_x \\ g_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ f \end{bmatrix}, \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\rangle_{\{x,y\}}.$$

497 Equation (SM3.10) follows from

$$\begin{aligned} \left\langle \begin{bmatrix} f_x \\ g_x \end{bmatrix} \right\rangle_{\{x\}} &= \left\langle \begin{bmatrix} x f_x \\ x g_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \mathbb{R} \left\{ \begin{bmatrix} f_x \\ g_x \end{bmatrix} \right\} \\ \left\langle \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\rangle_{\{x,y\}} &= \left\langle \begin{bmatrix} 0 \\ x g_y \end{bmatrix}, \begin{bmatrix} 0 \\ y g_y \end{bmatrix} \right\rangle_{\{x,y\}} \oplus \mathbb{R} \left\{ \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\}. \end{aligned}$$

498

499 We are now in a position to compute a universal unfolding of the normal form (5.5) using
 500 the analog of Theorem SM2.3.

501 *Proof of the Universal Unfolding Theorem 5.6.:* Compute

$$502 \quad \left\{ \begin{bmatrix} f_x(x) \\ g_x(x, y) \end{bmatrix}, \begin{bmatrix} 0 \\ g_y(x, y) \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_q \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t y \end{bmatrix} \right\}.$$

503 The tangent space is therefore

$$504 \quad T(F) = \begin{bmatrix} \mathcal{M}_x^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{x,y} \end{bmatrix} \oplus \mathbb{R} \left\{ \begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_q \end{bmatrix} \right\},$$

505 and a complement to $T(F)$ is a two dimensional complement to $(2\varepsilon_p x, \varepsilon_q)$ in the span

$$506 \quad \mathbb{R} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\}.$$

507 The complement

$$508 \quad \mathbb{R} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

509 leads to the universal unfolding (5.7).

510 **SM4. Hopf/Steady-State Mode Interaction: Proofs.** This section outlines proofs of
 511 the main results in Section 5.2 on Hopf/steady-state mode interaction. To apply singularity
 512 theory, we first use Liapunov-Schmidt reduction on the three-dimensional center manifold to
 513 construct a two-dimensional network whose zeros are in one-to-one correspondence with the
 514 equilibria and periodic solutions of the center manifold network. We compute the restricted
 515 tangent space for the Liapunov-Schmidt reduced network of the vector field (5.8), and use
 516 this result to prove Theorem 5.11, which states that (5.13) is a normal form. We then use the
 517 complement of the tangent space of (5.13) to identify the universal unfolding (5.14), proving
 518 Theorem 5.12.

519 **SM4.1. Liapunov-Schmidt Reduction for H/SS Mode Interaction.** In this subsection,
 520 we prove Theorem 5.8 using the standard ‘loop space’ approach to Hopf bifurcation via
 521 Liapunov-Schmidt reduction, Golubitsky and Schaeffer (1985) Chapter VIII. First we con-
 522 struct, from the center manifold vector field (5.8), an operator Φ with the property that
 523 solutions to $\Phi = 0$ correspond to periodic solutions of (5.8) with period approximately 2π .
 524 Then we apply Liapunov-Schmidt reduction to Φ to prove Theorem 5.8.

525 *Proof of Theorem 5.8.:* We seek periodic solutions of (5.8) with period approximately 2π ,
 526 for which we introduce τ corresponding to a rescaled time $s = t/(1 + \tau)$. In terms of s , (5.8)
 527 can be rewritten as

$$528 \quad (\text{SM4.1}) \quad \left[\begin{array}{c} \frac{dX}{ds} - (1 + \tau)f(X) \\ \frac{dy}{ds} - (1 + \tau)g(X, y) \end{array} \right] = 0.$$

529 A 2π -periodic solution of (SM4.1) corresponds to a periodic solution of (5.8) with period
 530 $2\pi(1 + \tau)$, which is close to 2π for $\tau \approx 0$. We define

$$531 \quad \Phi : \mathcal{C}_{2\pi}^1(\mathbb{R}^2) \times \mathcal{C}_{2\pi}^1(\mathbb{R}) \times \mathbb{R} \rightarrow \mathcal{C}_{2\pi}(\mathbb{R}^2) \times \mathcal{C}_{2\pi}(\mathbb{R}) \times \mathbb{R}$$

532 by the left hand side of (SM4.1). Then the zeros of $\Phi(X, y, \tau)$ characterizes periodic solutions
 533 of (5.8) with period approximately 2π .

534 We now apply the Liapunov-Schmidt reduction to Φ to find a reduced map ϕ in the
 535 coordinates (SM4.3) on $\ker(d\Phi)$. Then we use its properties to derive the theorem. The
 536 linearization of Φ about $(X, y, \tau) = (0, 0, 0)$ is

$$537 \quad (\text{SM4.2}) \quad d\Phi = \left[\begin{array}{cc} \frac{d}{ds} - \text{Df} & 0 \\ -\nabla_X g & \frac{d}{ds} \end{array} \right],$$

538
 539 whose kernel is 3-dimensional:

$$540 \quad (\text{SM4.3}) \quad \ker(d\Phi) = \left\langle \left[\begin{array}{c} c \\ -i(\nabla_X g) \cdot c \end{array} \right] e^{is}, \left[\begin{array}{c} \bar{c} \\ i(\nabla_X g) \cdot \bar{c} \end{array} \right] e^{-is}, \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \right\rangle.$$

541 Here c is a 2-dimensional complex eigenvector that satisfies $\text{Df} c = ic$. Identify $\ker(d\Phi)$ with
 542 \mathbb{R}^3 via the map

$$543 \quad (\text{SM4.4}) \quad (x_1, x_2, y) \rightarrow x_1 \text{Re}[we^{is}] + x_2 \text{Im}[we^{is}] + ye_3,$$

544 where $w = (c, -i(\nabla_u g) \cdot c)$ and $e_3 = (0, 0, 1)$. The circle group \mathbb{S}^1 acts on $\ker(d\Phi)$ by

$$545 \quad \gamma(\theta) = \begin{bmatrix} R(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

546 where $R(\theta)$ acts on \mathbb{R}^2 by rotation counterclockwise through the angle θ .

547 In the coordinates (SM4.4) on $\ker(d\Phi)$, the reduced map ϕ has the form

$$548 \quad \phi(x_1, x_2, y, \tau) = p(x_1^2 + x_2^2, \tau) \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + q(x_1^2 + x_2^2, \tau) \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} + \sigma(x_1^2 + x_2^2, y, \tau) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

549 because ϕ commutes with the action of \mathbb{S}^1 . Formulas for the derivatives of the reduced function
550 (Golubitsky and Schaeffer, 1985, p. 295) imply that $p(0, 0) = q(0, 0) = \sigma(0, 0, 0) = p_\tau(0, \tau) =$
551 $\sigma_\tau(0, 0, \tau) = 0$ and $q_\tau(0, \tau) = -1$. Solutions to $\phi = 0$ locally are in one-to-one correspondence
552 with periodic solutions of (5.8).

553 The rotational symmetry lets us assume that $x_2 = 0$, $x_1 \geq 0$, and the implicit function
554 theorem lets us solve $q(x_1^2, \tau) = 0$ for $\tau = \tau(x_1^2)$ (Golubitsky and Schaeffer, 1985, p. 345). Now
555 all solutions to $\phi = 0$ may be obtained from zeros of

$$556 \quad F(x_1, y) = \begin{bmatrix} r(x_1^2)x_1 \\ g(x_1^2, y) \end{bmatrix}$$

557 where $r(z) = p(z, \tau(z))$ and $g(z, y) = \sigma(z, y, \tau(z))$ and $x_1 \geq 0$.

558 **SM4.2. Restricted Tangent Space for H/SS Mode Interaction.** The restricted tan-
559 gent space of a map F in the context of symmetry, denoted by $RT(F)$, is obtained from
560 $\frac{d}{d\tau}\Gamma_\tau(F)|_{\tau=0}$, where Γ_τ is a one-parameter family of strong \mathbf{Z}_2 -equivalences (Definition 5.9)
561 with $\Gamma_0(F)(x, y) = F(x, y)$.

562 **Lemma SM4.1.** *Let $F = (r(u)x, g(u, y))$ be a map in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,y})$. A map $G \in (\mathcal{E}_u \cdot$
563 $\{x\}, \mathcal{E}_{u,y})$ lies in $RT(F)$ if and only if there exist maps $P_i(u) \in \mathcal{E}_u$ and $Q_j(u, y) \in \mathcal{E}_{u,y}$ such
564 that*

$$565 \quad G(x, y) = P_1 \begin{bmatrix} rx \\ 0 \end{bmatrix} + P_2 \begin{bmatrix} 2r_u ux + rx \\ 2g_u u \end{bmatrix} + Q_1 \begin{bmatrix} 0 \\ ru \end{bmatrix} + Q_2 \begin{bmatrix} 0 \\ g \end{bmatrix} + Q_3 \begin{bmatrix} 0 \\ ug_y \end{bmatrix} + Q_4 \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \blacksquare$$

567 *Proof.* The general form of strong \mathbf{Z}_2 -equivalence is (5.10). Define a one-parameter family
568 of strong \mathbf{Z}_2 -equivalences by

$$569 \quad (\text{SM4.5}) \quad \Gamma_\tau(F)(x, y) = \begin{bmatrix} a(u, \tau) & 0 \\ b(u, y, \tau)x & c(u, y, \tau) \end{bmatrix} \begin{bmatrix} r(\phi^2(u, \tau)u)\phi(u, \tau)x \\ g(\phi^2(u, \tau)u, \psi(u, y, \tau)) \end{bmatrix},$$

570 where Γ_0 is the identity and $\Gamma_\tau(F)(0, 0) = (0, 0)$. Then

(SM4.6)

$$571 \quad a(u, 0) = 1, b(u, y, 0) = 0, c(u, y, 0) = 1, \phi(u, 0) = 1, \psi(u, y, 0) = y, \psi(0, 0, \tau) = 0.$$

572 To compute the restricted tangent space, differentiate (SM4.5) with respect to τ and evaluate
 573 at $\tau = 0$, to obtain

$$574 \quad (SM4.7) \quad \begin{aligned} \dot{\Gamma}_0(F)(x, y) &= \dot{a}(u, 0) \begin{bmatrix} r(u)x \\ 0 \end{bmatrix} + \dot{b}(u, y, 0) \begin{bmatrix} 0 \\ r(u)u \end{bmatrix} + \dot{c}(u, y, 0) \begin{bmatrix} 0 \\ g(u, y) \end{bmatrix} \\ &+ \dot{\phi}(u, 0) \begin{bmatrix} 2r_u(u)ux + r(u)x \\ 2g_u(u, y)u \end{bmatrix} + \dot{\psi}(u, y, 0) \begin{bmatrix} 0 \\ g_y(u, y) \end{bmatrix}. \end{aligned}$$

575 Conditions (SM4.6) imply that $\dot{a}(u, 0)$, $\dot{b}(u, y, 0)$, $\dot{c}(u, y, 0)$ and $\dot{\phi}(u, 0)$ are arbitrary, whereas
 576 $\dot{\psi}(u, y, 0) = u\sigma(u, y) + y\nu(u, y)$ for arbitrary functions σ and ν . The restricted tangent space
 577 is therefore spanned by

$$578 \quad \left\langle \begin{bmatrix} rx \\ 0 \end{bmatrix}, \begin{bmatrix} 2r_u ux + rx \\ 2g_u u \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ ru \end{bmatrix}, \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ ug_y \end{bmatrix}, \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \right\rangle_{\{u, y\}}.$$

579 **SM4.3. Normal Form for H/SS Mode Interaction.** The proof of Theorem 5.11 is similar
 580 to the proof of Theorem 5.3, and is carried out in two steps. First, in Lemma SM4.2 we
 581 explicitly transform the lower order terms in $r(u)$ and $g(u, y)$ in $F = (r(u)x, g(u, y))$ into the
 582 normal form (5.13). Second, we use the tangent space constant theorem for the Hopf/steady-
 583 state mode interaction that is analogous to Theorem SM2.1 to transform away higher order
 584 terms.

585 The defining conditions for a Hopf/steady-state mode interaction imply that to first order
 586 in u and quadratic order in y , the functions $r(u)$ and $g(u, y)$ take the forms pu and $qu + ty^2$.
 587 Hence F takes the form $\bar{F} = (\bar{r}x, \bar{g})$ where

$$588 \quad (SM4.8) \quad \begin{aligned} \bar{r}(u)x &= pux \\ \bar{g}(u, y) &= qu + ty^2 \end{aligned}$$

589 **Lemma SM4.2.** Any map of the form (SM4.8) with $p, q, t \neq 0$ is strongly \mathbf{Z}_2 -equivalent to
 590 the normal form $\hat{F} = (\hat{r}x, \hat{g})$ where

$$591 \quad (SM4.9) \quad \begin{aligned} \hat{r}(u)x &= \varepsilon_p ux \\ \hat{g}(u, y) &= \varepsilon_q u + \varepsilon_t y^2 \end{aligned}$$

592 and $\varepsilon_* = \text{sign}(\ast)$.

593 *Proof.* We are interested only in terms of $r(u)$ and $g(u, y)$ up to first order in u and second
 594 order in y , so we compute the truncated forms of the transformation functions used to define
 595 \mathbf{Z}_2 -equivalence in (5.10), obtaining

$$596 \quad \phi(u) = \alpha \quad \psi(u, y) = \gamma y \quad a(u) = \delta \quad b(u, y) = 0 \quad c(u, y) = \rho.$$

597 Now

$$598 \quad (SM4.10) \quad \begin{bmatrix} \hat{r}(u)x \\ \hat{g}(u, y) \end{bmatrix} = \begin{bmatrix} \delta p \alpha^3 ux \\ \rho q \alpha^2 u + \rho t \gamma^2 y^2 \end{bmatrix}.$$

599 Combining like terms, the transformed coefficients are

$$600 \quad (\text{SM4.11}) \quad \begin{aligned} \hat{p} &= \delta \alpha^3 p \\ \hat{q} &= \rho \alpha^2 q \\ \hat{t} &= \rho \gamma^2 t. \end{aligned}$$

601 By assumption, $p \neq 0$, $q \neq 0$, $t \neq 0$ in \bar{r} and \bar{g} . Thus we can impose the conditions $\hat{p} = \varepsilon_p$,
602 $\hat{q} = \varepsilon_q$ and $\hat{t} = \varepsilon_t$, where $\varepsilon_* = \text{sign}(*)$, by making the transformation

$$603 \quad \begin{aligned} \delta &= \frac{\varepsilon_p}{p \alpha^3} \\ \rho &= \frac{\varepsilon_q}{\alpha^2 q} \\ \gamma^2 &= \frac{\alpha^2 q \varepsilon_t}{\varepsilon_q t}. \end{aligned}$$

604 Here $\alpha > 0$ is a free parameter. We require $\delta, \rho, \alpha, \gamma > 0$ to preserve the stabilities of steady
605 states. Were we free to choose the signs of δ, ρ , we could have transformed $\hat{p}, \hat{q}, \hat{t}$ to be $+1$.
606 Applying the above transformation to (SM4.8) produces (SM4.9).

607 *Proof of Theorem 5.11:* By Lemma SM4.2 a general map $F(x, y) = (r(u)x, g(u, y))$ sat-
608 isfying the defining and nondegeneracy conditions is strongly \mathbf{Z}_2 -equivalent to $\tilde{F} = \hat{F} + \dots$,
609 where \hat{F} is the normal form (5.13) and \dots indicates terms of higher order. Using the tan-
610 gent space constant theorem we remove higher order terms associated to \hat{F} by a suitable
611 transformation. Specifically, we show that $RT(\tilde{F}) = RT(\hat{F})$.

612 First, we claim that the restricted tangent space of the normal form $(\hat{r}x, \hat{g})$ is

$$613 \quad (\text{SM4.12}) \quad RT(\hat{F}) = \left[\begin{array}{c} \mathcal{M}_u \langle x \rangle \\ 0 \end{array} \right] \oplus \left[\begin{array}{c} 0 \\ \mathcal{M}_{u,y}^2 + \mathbb{R}\{u\} \end{array} \right],$$

614 which is a system of modules over the system of rings $(\mathcal{E}_u, \mathcal{E}_{u,y})$. By Lemma SM4.1, the
615 restricted tangent space for the normal form (5.13) is

$$616 \quad RT(\hat{F}) = \left\langle \left[\begin{array}{c} \varepsilon_p u x \\ 0 \end{array} \right], \left[\begin{array}{c} 3\varepsilon_p u x \\ 2\varepsilon_q u \end{array} \right] \right\rangle_{\{u\}} \oplus \left\langle \left[\begin{array}{c} 0 \\ \varepsilon_p u^2 \end{array} \right], \left[\begin{array}{c} 0 \\ \varepsilon_q u + \varepsilon_t y^2 \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t u y \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t y^2 \end{array} \right] \right\rangle_{\{u,y\}} \quad \blacksquare$$

617 Taking linear combinations, this reduces to

$$618 \quad RT(\hat{F}) = \left\langle \left[\begin{array}{c} u x \\ 0 \end{array} \right] \right\rangle_{\{u\}} \oplus \left\langle \left[\begin{array}{c} 0 \\ u \end{array} \right], \left[\begin{array}{c} 0 \\ y^2 \end{array} \right], \left[\begin{array}{c} 0 \\ u y \end{array} \right] \right\rangle_{\{u,y\}}.$$

619 The restricted tangent space is therefore given by (SM4.12).

620 Next, consider higher order maps $n \in \mathcal{M}_u^2$, and $m \in \mathcal{M}_{u,y}^3 \oplus \mathcal{M}_{u,y} \langle u \rangle$. That is, let
621 $\tilde{F} = (\tilde{r}x, \tilde{g}) = \hat{F} + (nx, m)$, where

$$622 \quad \begin{aligned} \tilde{r}(u)x &= (\varepsilon_p u + n(u))x \\ \tilde{g}(u, y) &= \varepsilon_q u + \varepsilon_t y^2 + m(u, y). \end{aligned}$$

623 We use Nakayama's Lemma to prove that $RT(\tilde{F}) = RT(\hat{F})$. Then $\tilde{F} = ((\hat{r} + n)x, \hat{g} + m)$ is
624 strongly \mathbf{Z}_2 -equivalent to the normal form \hat{F} , so F is strongly \mathbf{Z}_2 -equivalent to \hat{F} , as desired.

625 Observe that
 (SM4.13)

$$626 \quad RT(\tilde{F}) = \left\langle \left[\begin{array}{c} \varepsilon_p ux + nx \\ 0 \end{array} \right], \left[\begin{array}{c} 3\varepsilon_p ux + 2n_u ux + nx \\ 2\varepsilon_q u + 2m_u u \end{array} \right] \right\rangle_{\{u\}} \oplus \\
 \left\langle \left[\begin{array}{c} 0 \\ \varepsilon_p u^2 + nu \end{array} \right], \left[\begin{array}{c} 0 \\ \varepsilon_q u + \varepsilon_t y^2 + m \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t uy + um_y \end{array} \right], \left[\begin{array}{c} 0 \\ 2\varepsilon_t y^2 + ym_y \end{array} \right] \right\rangle_{\{u,y\}}.$$

627 By (SM4.12), each generator of $RT(\tilde{F})$ in (SM4.13) lies in $RT(\hat{F})$. Hence $RT(\tilde{F}) \subseteq RT(\hat{F})$.
 628 Next, we apply Nakayama's Lemma to prove that $RT(\hat{F}) \subseteq RT(\tilde{F})$, for which we need to
 629 show that $RT(\hat{F}) \subseteq RT(\tilde{F}) + (\mathcal{M}_u, \mathcal{M}_{u,y})RT(\hat{F})$. The generators of $RT(\hat{F})$ over the system
 630 of rings $(\mathcal{E}_u, \mathcal{E}_{u,y})$ are

$$631 \quad \left\{ \left[\begin{array}{c} ux \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ u \end{array} \right], \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] \right\}.$$

632 Therefore

$$633 \quad (\mathcal{M}_u, \mathcal{M}_{u,y})RT(\hat{F}) = \left\langle \left[\begin{array}{c} u^2 x \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ u^2 \end{array} \right], \left[\begin{array}{c} 0 \\ uy \end{array} \right], \left[\begin{array}{c} 0 \\ y^3 \end{array} \right] \right\rangle = \left[\begin{array}{c} \mathcal{M}_u^2 \langle x \rangle \\ \mathcal{M}_{u,y}^3 + \mathcal{M}_{u,y} \langle u \rangle \end{array} \right].$$

634 So we need to show that

$$635 \quad \left\langle \left[\begin{array}{c} ux \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ u \end{array} \right], \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] \right\rangle \subseteq RT(\tilde{F}) + \left\langle \left[\begin{array}{c} u^2 x \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ u^2 \end{array} \right], \left[\begin{array}{c} 0 \\ uy \end{array} \right], \left[\begin{array}{c} 0 \\ y^3 \end{array} \right] \right\rangle,$$

636 which follows from

$$637 \quad \begin{aligned} \left[\begin{array}{c} ux \\ 0 \end{array} \right] &= \left[\begin{array}{c} ux + nx \\ 0 \end{array} \right] - \left[\begin{array}{c} nx \\ 0 \end{array} \right] \in RT(\tilde{F}) + \left[\begin{array}{c} \mathcal{M}_u^2 \langle x \rangle \\ 0 \end{array} \right] \\ \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] &= \left[\begin{array}{c} 0 \\ y^2 + ym_y \end{array} \right] - \left[\begin{array}{c} 0 \\ ym_y \end{array} \right] \in RT(\tilde{F}) + \left[\begin{array}{c} 0 \\ \mathcal{M}_{u,y}^3 \oplus \mathcal{M}_{u,y} \langle u \rangle \end{array} \right] \\ \left[\begin{array}{c} 0 \\ u \end{array} \right] &= \left[\begin{array}{c} 0 \\ u + y^2 + m \end{array} \right] - \left[\begin{array}{c} 0 \\ y^2 \end{array} \right] - \left[\begin{array}{c} 0 \\ m \end{array} \right] \in RT(\tilde{F}) + \left[\begin{array}{c} 0 \\ \mathcal{M}_{u,y}^3 \oplus \mathcal{M}_{u,y} \langle u \rangle \end{array} \right]. \end{aligned}$$

638 Therefore the restricted tangent space of \tilde{F} is equal to the restricted tangent space of the
 639 original system F , so the two are equivalent.

640 **Remark SM4.3.** A complement in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,y})$ of the restricted tangent space of the
 641 normal form (5.13) is

$$642 \quad \mathbb{R} \left\{ \left[\begin{array}{c} x \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \end{array} \right], \left[\begin{array}{c} 0 \\ y \end{array} \right] \right\}$$

643 so the codimension of $RT(\hat{F})$ is 3.

644 **SM4.4. Tangent Space for H/SS Mode Interaction.** We follow the standard procedure
 645 used in previous cases, starting with

646 **Lemma SM4.4.** *The tangent space $T(F)$ of a map $F = (r(u)x, g(u, y))$ in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,y})$ is*

$$647 \quad (SM4.14) \quad T(F) = RT(F) \oplus \mathbb{R} \left\{ \left[\begin{array}{c} 0 \\ g_y(u, y) \end{array} \right] \right\}.$$

648 *Proof.* Computing the tangent space is similar to computing the restricted tangent space
 649 in Lemma SM4.1, except that now we do not require the origin to be fixed by the coordinate
 650 transformation. That is, $\dot{\psi}(u, y, 0)$ in (SM4.7) can be an arbitrary function. The tangent
 651 space is therefore

(SM4.15)

$$652 \left\langle \begin{bmatrix} r(u)x \\ 0 \end{bmatrix}, \begin{bmatrix} 2r_u(u)ux + r(u)x \\ 2g_u(u, y)u \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ r(u)u \end{bmatrix}, \begin{bmatrix} 0 \\ g(u, y) \end{bmatrix}, \begin{bmatrix} 0 \\ g_y(u, y) \end{bmatrix} \right\rangle_{\{u, y\}}.$$

653 Equation (SM4.14) follows from

$$\left\langle \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\rangle_{\{u, y\}} = \left\langle \begin{bmatrix} 0 \\ ug_y \end{bmatrix}, \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \right\rangle_{\{u, y\}} \oplus \mathbb{R} \left\{ \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\}.$$

654

655 We now find a universal unfolding of the normal form (5.13) according to the analog of
 656 Theorem SM2.3.

657 *Proof of the Universal Unfolding Theorem 5.12.:* Compute

$$658 \text{ (SM4.16)} \quad \begin{bmatrix} 0 \\ g_y(x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 2\varepsilon_t y \end{bmatrix}.$$

659 The tangent space is therefore

$$660 T(F) = \begin{bmatrix} \mathcal{M}_u \langle x \rangle \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{u, y} \end{bmatrix},$$

661 and the complement to $T(F)$ is two dimensional:

$$662 \mathbb{R} \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

663 This leads to the universal unfolding (5.14).

664 **SM5. Steady-State/Hopf Mode Interaction: Proofs.** This section outlines proofs for
 665 the main results presented in Section 5.3 for steady-state/Hopf mode interaction. The basic
 666 strategy is the same as in Section SM4, though some differences appear in the details of the
 667 proofs. In order to apply the methods of singularity theory, we use Liapunov-Schmidt reduc-
 668 tion on the three-dimensional center manifold to construct a two-dimensional network whose
 669 zeros are in one-to-one correspondence with the equilibria and periodic solutions of the center
 670 manifold network. We compute the restricted tangent space for this reduced network for the
 671 vector field (5.15), and use this to prove Theorem 5.17 that (5.18) is a normal form. We then
 672 use the complement of the tangent space of (5.18) to identify the universal unfolding (5.19),
 673 proving Theorem 5.18.

674 **SM5.1. Liapunov-Schmidt Reduction for SS/H Mode Interaction.** We outline a proof
 675 of Theorem 5.14. Begin with the system (5.15) that describes the center manifold dynamics
 676 for the steady-state Hopf mode interaction. Assume that the origin is an equilibrium, so that

677 $f(0) = g(0, 0) = 0$; that f is associated with steady-state bifurcation, so that $f_x(0) = 0$; and
 678 that a Hopf bifurcation is associated with g , so that $D_Y g$ has eigenvalues $\pm i$ at the origin.
 679 We seek periodic solutions and rescale time by $t = (1 + \tau)s$ to set the period to 2π . We can
 680 now define a map

$$681 \quad \Phi : \mathcal{C}_{2\pi}^1(\mathbb{R}) \times \mathcal{C}_{2\pi}^1(\mathbb{R}^2) \times \mathbb{R} \rightarrow \mathcal{C}_{2\pi}(\mathbb{R}) \times \mathcal{C}_{2\pi}(\mathbb{R}^2) \times \mathbb{R}$$

682 given by

$$683 \quad \begin{aligned} \Phi_1(u, v, \tau) &= \frac{du}{ds} - (1 + \tau)f(u) \\ \Phi_2(u, v, \tau) &= \frac{dv}{ds} - (1 + \tau)g(u, v) \end{aligned}$$

684 where $u \in \mathcal{C}_{2\pi}^1(\mathbb{R})$ and $v \in \mathcal{C}_{2\pi}^1(\mathbb{R}^2)$ are once-differentiable 2π -periodic functions on \mathbb{R} and \mathbb{R}^2
 685 respectively, and $\tau \in \mathbb{R}$.

686 The zeros of Φ correspond to periodic solutions for the center manifold vector field. The
 687 linearization of Φ about $(u, v, \tau) = (0, 0, 0)$ is

$$688 \quad d\Phi = \begin{bmatrix} \frac{d}{ds} & 0 \\ -g_u & \frac{d}{ds} - D_v g \end{bmatrix}$$

689 and an element of the kernel, $\eta(s) \in \ker(d\Phi)$, is

$$690 \quad \eta(s) = x \begin{bmatrix} 1 \\ (D_v g)^{-1} g_u \end{bmatrix} + \operatorname{Re} \left[z \begin{bmatrix} 0 \\ c \end{bmatrix} e^{is} \right],$$

691 for coordinates $x \in \mathbb{R}$ and $z \in \mathbb{C}$, where $c \in \mathbb{C}^2$ is the complex eigenvector defined by
 692 $(D_v g)c = ic$. Moreover, the \mathbb{S}^1 -action is

$$693 \quad \gamma(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

694 The reduction now proceeds as in standard Hopf bifurcation, [Golubitsky and Schaeffer \(1985\)](#).
 695 Periodic solutions of the vector field are locally in one-to-one correspondence with zeros of the
 696 function $F(x, y) = (f(x), r(x, y^2)y)$ on \mathbb{R}^2 .

697 **SM5.2. Restricted Tangent Space for SS/H Mode Interaction.** The proof of Theo-
 698 rem [5.17](#) requires computing the restricted tangent space $RT(F)$ of the map [\(5.18\)](#). Let Γ_τ
 699 be a one-parameter family of strong \mathbf{Z}_2 -equivalences as in [\(5.15\)](#), with $\Gamma_0(F)(x, y) = F(x, y)$.
 700 A typical element of $RT(F)$ is

$$701 \quad \left. \frac{d}{d\tau} \Gamma_\tau(F) \right|_{\tau=0}$$

702 The analog of the tangent space constant theorem, Theorem [SM2.1](#), lets us prove equivalence
 703 of maps. Here we use restricted tangent spaces in the \mathbf{Z}_2 -symmetric context.

704 **Lemma SM5.1.** *Let $F = (f(x), r(x, v)y)$ be a map in $(\mathcal{E}_x, \mathcal{E}_{x,v} \cdot \{y\})$. A map $G \in (\mathcal{E}_x, \mathcal{E}_{x,v} \cdot$
705 $\{y\})$ is in $RT(F)$ if and only if there exist maps $P_i(x) \in \mathcal{E}_x$ and $Q_j(x, v) \in \mathcal{E}_{x,v}$ such that*

$$706 \quad G(x, y) = P_1 \begin{bmatrix} f \\ 0 \end{bmatrix} + P_2 \begin{bmatrix} xf_x \\ xy r_x \end{bmatrix} + Q_1 \begin{bmatrix} 0 \\ yf \end{bmatrix} + Q_2 \begin{bmatrix} 0 \\ yr \end{bmatrix} + Q_3 \begin{bmatrix} 0 \\ yvr_v \end{bmatrix}. \quad 707$$

708 *Proof.* Define a parametrized family of near-identity transformations (5.17), generating
709 an orbit of strongly equivalent systems near the original vector field $F(x, y)$ for all small τ , by

$$710 \quad (\text{SM5.1}) \quad \Gamma_\tau(F)(x, y) = \begin{bmatrix} a(x, \tau) & 0 \\ b(x, v, \tau)y & c(x, v, \tau) \end{bmatrix} \begin{bmatrix} f(\phi(x, \tau)) \\ r(\phi(x, \tau), \psi(x, v, \tau)^2 v) \psi(x, v, \tau)y \end{bmatrix},$$

711 where Γ_0 is the identity and $\Gamma_\tau(F)(0, 0) = (0, 0)$. Then

$$712 \quad (\text{SM5.2}) \quad a(x, 0) = 1 \quad b(x, v, 0) = 0 \quad c(x, v, 0) = 1 \quad \phi(x, 0) = x \quad \psi(x, v, 0) = 1.$$

713 Compute the restricted tangent space by differentiating (SM5.1):

$$714 \quad \dot{\Gamma}_0(F)(x, y) = \dot{a}(x, 0) \begin{bmatrix} f(x) \\ 0 \end{bmatrix} + \dot{b}(x, v, 0) \begin{bmatrix} 0 \\ f(x)y \end{bmatrix} + \dot{c}(x, v, 0) \begin{bmatrix} 0 \\ r(x, v)y \end{bmatrix} \\ + \dot{\chi}(x, 0) \begin{bmatrix} f_x(x)x \\ r_x(x, v)xy \end{bmatrix} + \dot{\psi}(x, v, 0) \begin{bmatrix} 0 \\ (2r_v(x, v)v + r(x, v))y \end{bmatrix},$$

715 where $\phi(x, 0) = \chi(x, 0)x$ and $\chi(x, 0) = 1$. Equations (SM5.2) imply that

$$716 \quad \dot{a}(x, 0) \quad \dot{b}(x, v, 0) \quad \dot{c}(x, v, 0) \quad \dot{\chi}(x, 0) \quad \dot{\psi}(x, v, 0)$$

717 are arbitrary functions. The restricted tangent space is therefore spanned by

$$718 \quad \left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} xf_x \\ yxr_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ yf \end{bmatrix}, \begin{bmatrix} 0 \\ yr \end{bmatrix}, \begin{bmatrix} 0 \\ yvr_v \end{bmatrix} \right\rangle_{\{x, v\}}.$$

719 **SM5.3. Normal Form for SS/H Mode Interaction.** We prove that (5.18) is a normal
720 form by showing that a given admissible \mathbf{Z}_2 -equivariant F of the form (5.16) satisfying the
721 defining and nondegeneracy conditions of Theorem 5.17 can be transformed to (5.18) via a
722 transformation of the form (5.17). Consider a map $\bar{F}(x, y) = (\bar{f}(x), \bar{r}(x, v)y)$, where

$$723 \quad (\text{SM5.3}) \quad \begin{aligned} \bar{f}(x) &= px^2 \\ \bar{r}(x, v)y &= (qv + sx)y, \end{aligned}$$

724 and $v = y^2$. Apply the tangent space constant theorem to show that all other nonlin-
725 ear terms can be removed by a suitable transformation. Specifically, we compute $RT(\bar{F})$
726 in Lemma SM5.2 and then show that $RT(F) = RT(\bar{F})$ for the given F . Finally, by an
727 appropriate (orientation preserving) rescaling of \bar{F} , we obtain the normal form (5.18).

728 **Lemma SM5.2.** *The restricted tangent space of $(\bar{f}, \bar{r}y)$ given by (SM5.3) is*

$$729 \quad (\text{SM5.4}) \quad RT(\bar{F}) = \begin{bmatrix} \mathcal{M}_x^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{x, v}\langle y \rangle \end{bmatrix},$$

730 *which is a system of modules over the system of rings $(\mathcal{E}_x, \mathcal{E}_{x, v})$.*

731 *Proof.* By Lemma SM5.1 the restricted tangent space for (SM5.3) is

$$732 \quad RT(\bar{F}) = \left\langle \begin{bmatrix} px^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2px^2 \\ sxy \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ px^2y \end{bmatrix}, \begin{bmatrix} 0 \\ (qv + sx)y \end{bmatrix}, \begin{bmatrix} 0 \\ (3qv + sx)y \end{bmatrix} \right\rangle_{\{x,v\}}.$$

733 Taking linear combinations, this reduces to

$$734 \quad RT(\bar{F}) = \left\langle \begin{bmatrix} x^2 \\ 0 \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\rangle_{\{x,v\}}.$$

735 The restricted tangent space is therefore (SM5.4).

736 *Proof of Theorem 5.17:* The restricted tangent space of (SM5.3) is given by Lemma SM5.2. ■
 737 We show that a general \mathbf{Z}_2 -equivariant map $\tilde{F} = \bar{F} + \dots$, where \bar{F} is given by (SM5.3), and
 738 \dots indicates higher-order admissible perturbations. We use Nakayama's Lemma to show that
 739 $RT(\tilde{F}) = RT(\bar{F})$, so the tangent space constant theorem guarantees that the two maps \tilde{F}
 740 and \bar{F} are strongly \mathbf{Z}_2 -equivalent as in Definition 5.15.

741 Let $\tilde{F}(x, y) = (\tilde{f}(x), \tilde{r}(x, v)y)$, where

$$742 \quad \begin{aligned} \tilde{f}(x) &= px^2 + n(x) \\ \tilde{r}(x, v)y &= (qv + sx + m(x, v))y, \end{aligned}$$

743 and $n \in \mathcal{M}_x^3$, $m \in \mathcal{M}_{x,v}^2$. Lemma SM5.1 shows that

$$744 \quad \text{(SM5.5)} \quad RT(\tilde{F}) = \left\langle \begin{bmatrix} px^2 + n \\ 0 \end{bmatrix}, \begin{bmatrix} 2px^2 + xn_x \\ (s + m_x)xy \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ (px^2 + n)y \end{bmatrix}, \begin{bmatrix} 0 \\ (qv + sx + m)y \end{bmatrix}, \begin{bmatrix} 0 \\ (3qv + sx + 2vm_v + m)y \end{bmatrix} \right\rangle_{\{x,v\}}.$$

745 By Lemma SM5.2, each generator of $RT(\tilde{F})$ in (SM5.5).

746 lies in $RT(\bar{F})$, so $RT(\tilde{F}) \subseteq RT(\bar{F})$. Next we apply Nakayama's Lemma to prove $RT(\bar{F}) \subseteq$
 747 $RT(\tilde{F})$, for which we need to show $RT(\bar{F}) \subseteq RT(\tilde{F}) + (\mathcal{M}_x, \mathcal{M}_{x,v})RT(\bar{F})$. A set of generators
 748 of $RT(\bar{F})$ over the system of rings $(\mathcal{E}_x, \mathcal{E}_{x,v})$ is

$$749 \quad \left\{ \begin{bmatrix} x^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\}.$$

750 Therefore

$$751 \quad (\mathcal{M}_x, \mathcal{M}_{x,v})RT(\bar{F}) = \left\langle \begin{bmatrix} x^3 \\ 0 \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ xvy \end{bmatrix}, \begin{bmatrix} 0 \\ v^2y \end{bmatrix}, \begin{bmatrix} 0 \\ x^2y \end{bmatrix} \right\rangle_{\{x,v\}} = \begin{bmatrix} \mathcal{M}_x^3 \\ \mathcal{M}_{x,v}^2 \langle y \rangle \end{bmatrix},$$

752 and we need to show that

$$753 \quad \left\langle \begin{bmatrix} x^2 \\ 0 \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\rangle_{\{x,v\}} \subseteq RT(\tilde{F}) + \begin{bmatrix} \mathcal{M}_x^3 \\ \mathcal{M}_{x,v}^2 \langle y \rangle \end{bmatrix}.$$

754 This follows from

$$\begin{aligned}
\begin{bmatrix} x^2 \\ 0 \end{bmatrix} &= \frac{1}{p} \begin{bmatrix} px^2 + n \\ 0 \end{bmatrix} - \frac{1}{p} \begin{bmatrix} n \\ 0 \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} \mathcal{M}_x^3 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ vy \end{bmatrix} &= \frac{1}{2q} \left(\begin{bmatrix} 0 \\ (3qv + sx + 2vm_v + m)y \end{bmatrix} - \begin{bmatrix} 0 \\ (qv + sx + m)y \end{bmatrix} \right) \\
755 \quad &- \frac{1}{q} \begin{bmatrix} 0 \\ vym_v \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{x,v}^2 \langle y \rangle \end{bmatrix} \\
\begin{bmatrix} 0 \\ xy \end{bmatrix} &= \frac{1}{2s} \left(3 \begin{bmatrix} 0 \\ (qv + sx + m)y \end{bmatrix} - \begin{bmatrix} 0 \\ (3qv + sx + 2vm_v + m)y \end{bmatrix} \right) \\
&- \frac{1}{s} \begin{bmatrix} 0 \\ ym - yvm_v \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{x,v}^2 \langle y \rangle \end{bmatrix}.
\end{aligned}$$

756 Therefore the restricted tangent space of \tilde{F} is equal to the restricted tangent space of \bar{F} , so
757 \tilde{F} and \bar{F} are equivalent. Moreover the transformation $f \rightarrow f/|p|$, $r \rightarrow r/|q|$ and $x \rightarrow |q|x/|s|$
758 takes \bar{F} to the normal form (5.18).

759 **Remark SM5.3.** The restricted tangent space has finite codimension. The complement of
760 the restricted tangent space of the normal form (5.18) in $(\mathcal{E}_x, \mathcal{E}_{x,v} \cdot \{y\})$ is

$$761 \quad \mathbb{R} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\},$$

762 so the codimension of $RT(\hat{F})$ is 3.

763 **SM5.4. Tangent Space for SS/H Mode Interaction.** As usual we first specify the rele-
764 vant tangent space:

765 **Lemma SM5.4.** *The tangent space $T(F)$ of a map $F = (f(x), r(x, v)y)$ in $(\mathcal{E}_x, \mathcal{E}_{x,v} \cdot \{y\})$ is*

$$766 \quad (\text{SM5.6}) \quad T(F) = RT(F) \oplus \mathbb{R} \left\{ \begin{bmatrix} f_x \\ yr_x \end{bmatrix} \right\}.$$

767 *Proof.* Computing the tangent space is similar to computing the restricted tangent space
768 in Lemma SM4.1, except that now we do not require the origin to be fixed by the coordinate
769 transformation. This means that we no longer enforce $\dot{\phi}(x, 0) = \dot{\chi}(x, 0)x$, and instead take
770 $\dot{\phi}(x, 0)$ to be an arbitrary function. Hence the tangent space is

$$771 \quad \left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} f_x \\ yr_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ yf \end{bmatrix}, \begin{bmatrix} 0 \\ yr \end{bmatrix}, \begin{bmatrix} 0 \\ y(2vr_v + r) \end{bmatrix} \right\rangle_{\{x,v\}}.$$

772 Relaxing the restriction of fixing the origin modifies the second element of the first span
773 compared to the calculation for the restricted tangent space. In fact we can write the span of

774 this modified vector in terms of a span of vectors of $RT(F)$ as

$$775 \quad \left\langle \begin{bmatrix} f_x \\ yr_x \end{bmatrix} \right\rangle_{\{x\}} = \left\langle \begin{bmatrix} xf_x \\ xy r_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \mathbb{R} \left\{ \begin{bmatrix} f_x \\ yr_x \end{bmatrix} \right\}.$$

776 The tangent space is therefore given by (SM5.6).

777 By computing the complement of $T(\hat{F})$, we derive a universal unfolding of the normal
778 form (5.18) using the analog of Theorem SM2.3.

779 *Proof of the Universal Unfolding Theorem 5.18:* The restricted tangent space $RT(\hat{F}) =$
780 $RT(\bar{F})$ is given by Lemma SM5.2. We must therefore compute

$$781 \quad \begin{bmatrix} \hat{f}_x(x) \\ \hat{r}_x(x, v)y \end{bmatrix} = \begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_s y \end{bmatrix}.$$

782 The tangent space is

$$783 \quad T(F) = \begin{bmatrix} \mathcal{M}_x^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{x,v} \langle y \rangle \end{bmatrix} \oplus \mathbb{R} \left\{ \begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_s y \end{bmatrix} \right\}.$$

784 A two-dimensional complement of $T(F)$ can be spanned either by

$$785 \quad \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle$$

786 or by

$$787 \quad \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle.$$

788 A universal unfolding corresponding to the latter choice is (5.19).

789 **SM6. Hopf/Hopf Mode Interaction: Proofs.** This section outlines proofs for the main
790 results presented in Section 5.4 for Hopf/Hopf mode interaction. In order to apply the methods
791 of singularity theory, we assume the four-dimensional center manifold dynamics is in Birkhoff
792 normal form. We can then reduce it to the dynamics of a two-dimensional network, whose
793 vector field (5.22) commutes with the standard action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ in the plane. We compute
794 the restricted tangent space for this reduced network, and use the result to prove Theorem
795 5.24, which states that (5.24) is a normal form.

796 **SM6.1. Amplitude Reduction for H/H Mode Interaction.** Here we outline a proof of
797 Theorem 5.22. Begin with the system (5.20) that describes the center manifold dynamics for
798 the Hopf/Hopf mode interaction. Assume that the origin is an equilibrium so that $f(0) =$
799 $g(0, 0) = 0$, and that the linear part of (f, g) is nonresonant, so that $D_X f$ and $D_Y g$ have two
800 distinct pairs of complex conjugate purely imaginary eigenvalues $\pm i\omega$ and $\pm i\nu$ at the origin,
801 with ω and ν irrationally related. Assume also that (5.20) is in Birkhoff normal form, so that
802 (f, g) commutes with the two-torus \mathbb{T}^2 whose action on \mathbb{R}^4 is (5.21). Equivalently, \mathbb{T}^2 acts on

803 \mathbb{C}^2 by

$$804 \quad (\text{SM6.1}) \quad (\psi_1, \psi_2)(z_1, z_2) = (e^{i\psi_1} z_1, e^{i\psi_2} z_2)$$

805 where $(\psi_1, \psi_2) \in \mathbb{T}^2$ and $(z_1, z_2) \in \mathbb{C}^2$. Now

$$806 \quad (\psi_1, \psi_2)(f(z_1), g(z_1, z_2)) = (f(e^{i\psi_1} z_1), g(e^{i\psi_1} z_1, e^{i\psi_2} z_2)),$$

807 which implies

$$808 \quad (\text{SM6.2}) \quad (f(z_1), g(z_1, z_2)) = (P_1(|z_1|^2)z_1, P_2(|z_1|^2, |z_2|^2)z_2) ,$$

809 where $P_1(0) = \omega i$ and $P_2(0, 0) = \nu i$.

810 Set $z_1 = x e^{i\theta_1}$ and $z_2 = y e^{i\theta_2}$. Using (SM6.2) we can reduce the Birkhoff normal form
811 (f, g) to the amplitude equations (5.22), which we write as

$$812 \quad F(x, y) = (p_1(x^2)x, p_2(x^2, y^2)y) .$$

813 Here p_j is the real part of P_j for $j = 1, 2$, so that $p_1(0) = p_2(0, 0) = 0$.

814 **SM6.2. Restricted Tangent Space for H/H Mode Interaction.** The proof of Theo-
815 rem 5.24 requires computing the restricted tangent space $RT(F)$ of the map F in (5.24).
816 Let Γ_τ be a one-parameter family of strong $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ -equivalences (as in Definition 5.23) with
817 $\Gamma_0(F)(x, y) = F(x, y)$. A typical element of $RT(F)$ is

$$818 \quad \left. \frac{d}{d\tau} \Gamma_\tau(F) \right|_{\tau=0} .$$

819 The analog of the tangent space constant theorem, Theorem SM2.1, lets us prove equivalence
820 of the relevant maps. The restricted tangent spaces involved are computed in the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ -
821 symmetric context using Lemma SM6.1.

822 **Lemma SM6.1.** *Let $F = (r(u)x, s(u, v)y)$ be a map in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$. A map $G \in$
823 $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is in $RT(F)$ if and only if there exist maps $P_i(u) \in \mathcal{E}_u$ and $Q_j(u, v) \in \mathcal{E}_{u,v}$
824 such that*

$$825 \quad G(x, y) = P_1 \begin{bmatrix} xr \\ 0 \end{bmatrix} + P_2 \begin{bmatrix} xr_u u \\ ys_u u \end{bmatrix} + Q_1 \begin{bmatrix} 0 \\ ys \end{bmatrix} + Q_2 \begin{bmatrix} 0 \\ yru \end{bmatrix} + Q_3 \begin{bmatrix} 0 \\ ys_v v \end{bmatrix} .$$

827 *Proof.* Define a parametrized family of near-identity transformations (5.23), generating
828 an orbit of strongly equivalent systems near the original vector field $F(x, y)$ for all small τ , by
(SM6.3)

$$829 \quad \Gamma_\tau(F)(x, y) = \begin{bmatrix} a(u, \tau) & 0 \\ b(u, v, \tau)xy & c(u, v, \tau) \end{bmatrix} \begin{bmatrix} r(\phi^2(u, \tau)u)\phi(u, \tau)x \\ s(\phi^2(u, \tau)u, \psi^2(u, v, \tau)v)\psi(u, v, \tau)y \end{bmatrix} ,$$

830 where Γ_0 is the identity and $\Gamma_\tau(F)(0, 0) = (0, 0)$. Then

$$831 \quad (\text{SM6.4}) \quad a(u, 0) = 1 \quad b(u, v, 0) = 0 \quad c(u, v, 0) = 1 \quad \phi(u, 0) = 1 \quad \psi(u, v, 0) = 1$$

832 Compute the restricted tangent space by differentiating (SM6.3):

$$833 \quad \dot{\Gamma}_0(F)(x, y) = \dot{a}(u, 0) \begin{bmatrix} r(u)x \\ 0 \end{bmatrix} + \dot{b}(u, v, 0) \begin{bmatrix} 0 \\ r(u)uy \end{bmatrix} + \dot{c}(u, v, 0) \begin{bmatrix} 0 \\ s(u, v)y \end{bmatrix} \\ + \dot{\phi}(u, 0) \begin{bmatrix} (2r_u(u)u + r(u))x \\ 2s_u(u, v)uy \end{bmatrix} + \dot{\psi}(u, v, 0) \begin{bmatrix} 0 \\ (2s_v(x, v)v + s(u, v))y \end{bmatrix}.$$

834 Equations (SM6.4) imply that

$$835 \quad \dot{a}(u, 0) \quad \dot{b}(u, v, 0) \quad \dot{c}(u, v, 0) \quad \dot{\phi}(u, 0) \quad \dot{\psi}(u, v, 0)$$

836 are arbitrary functions. The restricted tangent space is therefore spanned by

$$837 \quad \left\langle \begin{bmatrix} xr \\ 0 \end{bmatrix}, \begin{bmatrix} xr_u u \\ ys_u u \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ ys \end{bmatrix}, \begin{bmatrix} 0 \\ yru \end{bmatrix}, \begin{bmatrix} 0 \\ ys_v v \end{bmatrix} \right\rangle_{\{u, v\}}.$$

838 **SM6.3. Normal form for H/H Mode Interaction.** We prove that (5.24) is a normal form
839 by showing that a given admissible $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ -equivariant F of the form (5.22), satisfying the
840 defining and nondegeneracy conditions of Theorem 5.24, can be transformed to (5.24) via a
841 transformation of the form (5.23).

842 The defining conditions for Hopf/Hopf mode interaction imply that to first order in u and
843 v , the functions $r(u)$ and $s(u, v)$ take the forms pu and $qu + tv$. Therefore F takes the form
844 $\bar{F}(x, y) = (\bar{r}(u)x, \bar{s}(u, v)y)$ with

$$845 \quad (\text{SM6.5}) \quad \begin{aligned} \bar{r}(u)x &= pux \\ \bar{s}(u, v)y &= (qu + tv)y, \end{aligned}$$

846 where $u = x^2$, $v = y^2$. We prove Theorem 5.24 in two steps. First, we apply the tangent space
847 constant theorem to transform away all other higher order terms by showing that $RT(F) =$
848 $RT(\bar{F})$. Second, we rescale \bar{F} to obtain the normal form (5.24).

849 **Lemma SM6.2.** *The restricted tangent space of $(\bar{r}x, \bar{s}y)$ given by (SM6.5) is*

$$850 \quad RT(\bar{F}) = \begin{bmatrix} \mathcal{M}_u \langle x \rangle \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{u, v} \langle y \rangle \end{bmatrix},$$

851 *which is a system of modules over the system of rings $(\mathcal{E}_u, \mathcal{E}_{u, v})$.*

852 *Proof.* By Lemma SM6.1 the restricted tangent space for (SM6.5) is

$$853 \quad RT(\bar{F}) = \left\langle \begin{bmatrix} xpu \\ 0 \end{bmatrix}, \begin{bmatrix} xpu \\ yqu \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ y(qu + tv) \end{bmatrix}, \begin{bmatrix} 0 \\ ypu^2 \end{bmatrix}, \begin{bmatrix} 0 \\ ytv \end{bmatrix} \right\rangle_{\{u, v\}}.$$

854 Taking linear combinations, this reduces to

$$855 \quad RT(\bar{F}) = \left\langle \begin{bmatrix} xu \\ 0 \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ yv \end{bmatrix}, \begin{bmatrix} 0 \\ yu \end{bmatrix} \right\rangle_{\{u, v\}}.$$

856 The restricted tangent space is therefore given by (SM6.2).

857 *Proof of Theorem 5.24:* Now $F = \bar{F} + \dots$ where \bar{F} is given by (SM6.5) and \dots indicates
 858 admissible higher-order perturbations. We use Nakayama's Lemma to show that $RT(F) =$
 859 $RT(\bar{F})$, so the tangent space constant theorem guarantees that the maps F and \bar{F} are strongly
 860 $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ -equivalent, as in Definition 5.23. Then we rescale \bar{F} to obtain the normal form (5.24).
 861 We can write $F(x, y) = (r(u)x, s(u, v)y)$ as

$$862 \quad \begin{aligned} r(u) &= (pu + n(u))x \\ s(u, v)y &= (qu + tv + m(u, v))y, \end{aligned}$$

863 where $n \in \mathcal{M}_u^2$, $m \in \mathcal{M}_{u,v}^2$ are higher order maps. By Lemma SM6.1,

$$864 \quad (SM6.6) \quad RT(F) = \left\langle \left[\begin{array}{c} x(pu + n) \\ 0 \end{array} \right], \left[\begin{array}{c} x(p + n_u)u \\ y(q + m_u)u \end{array} \right] \right\rangle_{\{u\}} \oplus \\ \left\langle \left[\begin{array}{c} 0 \\ y(qu + tv + m) \end{array} \right], \left[\begin{array}{c} 0 \\ y(pu + n)u \end{array} \right], \left[\begin{array}{c} 0 \\ y(t + m_v)v \end{array} \right] \right\rangle_{\{u,v\}}.$$

865 By Lemma SM6.2, each generator of $RT(F)$ in (SM6.6) is in $RT(\bar{F})$, so $RT(F) \subseteq RT(\bar{F})$.
 866 Next we apply Nakayama's Lemma to prove $RT(\bar{F}) \subseteq RT(F)$, for which we need to show
 867 $RT(\bar{F}) \subseteq RT(F) + (\mathcal{M}_u, \mathcal{M}_{u,v})RT(\bar{F})$. The set of generators of $RT(\bar{F})$ over the system of
 868 rings $(\mathcal{E}_u, \mathcal{E}_{u,v})$ is

$$869 \quad \left\{ \left[\begin{array}{c} ux \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ vy \end{array} \right], \left[\begin{array}{c} 0 \\ uy \end{array} \right] \right\}.$$

870 Therefore

$$871 \quad (\mathcal{M}_u, \mathcal{M}_{u,v})RT(\bar{F}) = \left\langle \left[\begin{array}{c} u^2x \\ 0 \end{array} \right] \right\rangle_{\{u\}} \oplus \left\langle \left[\begin{array}{c} 0 \\ uv y \end{array} \right], \left[\begin{array}{c} 0 \\ v^2y \end{array} \right], \left[\begin{array}{c} 0 \\ u^2y \end{array} \right] \right\rangle_{\{u,v\}} = \left[\begin{array}{c} \mathcal{M}_u^2 \langle x \rangle \\ \mathcal{M}_{u,v}^2 \langle y \rangle \end{array} \right].$$

872 Now we must show that

$$873 \quad \left\langle \left[\begin{array}{c} ux \\ 0 \end{array} \right] \right\rangle_{\{u\}} \oplus \left\langle \left[\begin{array}{c} 0 \\ vy \end{array} \right], \left[\begin{array}{c} 0 \\ uy \end{array} \right] \right\rangle_{\{u,v\}} \subseteq RT(F) + \left[\begin{array}{c} \mathcal{M}_u^2 \langle x \rangle \\ \mathcal{M}_{u,v}^2 \langle y \rangle \end{array} \right].$$

874 This follows from

$$875 \quad \begin{aligned} \left[\begin{array}{c} ux \\ 0 \end{array} \right] &= \frac{1}{p} \left[\begin{array}{c} x(pu + n) \\ 0 \end{array} \right] - \frac{1}{p} \left[\begin{array}{c} nx \\ 0 \end{array} \right] \in RT(F) + \left[\begin{array}{c} \mathcal{M}_u^2 \langle x \rangle \\ 0 \end{array} \right] \\ \left[\begin{array}{c} 0 \\ vy \end{array} \right] &= \frac{1}{t} \left[\begin{array}{c} 0 \\ y(t + m_v)v \end{array} \right] - \frac{1}{t} \left[\begin{array}{c} 0 \\ yv m_v \end{array} \right] \in RT(F) + \left[\begin{array}{c} 0 \\ \mathcal{M}_{u,v}^2 \langle y \rangle \end{array} \right] \\ \left[\begin{array}{c} 0 \\ uy \end{array} \right] &= \frac{1}{q} \left(\left[\begin{array}{c} 0 \\ y(qu + tv + m) \end{array} \right] - \left[\begin{array}{c} 0 \\ y(t + m_v)v \end{array} \right] \right) - \frac{1}{q} \left[\begin{array}{c} 0 \\ ym - yv m_v \end{array} \right] \\ &\in RT(F) + \left[\begin{array}{c} 0 \\ \mathcal{M}_{u,v}^2 \langle y \rangle \end{array} \right]. \end{aligned}$$

876 Therefore the restricted tangent space of F is equal to the restricted tangent space of \bar{F} , so
 877 the two are equivalent. Moreover the transformation $r \rightarrow r/|p|$, $s \rightarrow s/|q|$ and $v \rightarrow |q|v/|t|$
 878 takes \bar{F} to the normal form (5.24).

879 **Remark SM6.3.** The restricted tangent space has finite codimension. A complement of the
 880 restricted tangent space of the normal form (5.24) in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is

$$881 \quad \mathbb{R} \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\}.$$

882 The codimension of $RT(\hat{F})$ is 2.

883 **SM6.4. Tangent Space for H/H Mode Interaction.** We find a universal unfolding in
 884 terms of the complement of the tangent space, which is forced to be identical to the restricted
 885 tangent space by the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ -symmetry.

886 **Lemma SM6.4.** *The tangent space $T(F)$ of $F = (r(u)x, s(u, v)y)$ in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is*
 887 *equal to $RT(F)$.*

888 *Proof.* Computing $T(F)$ is similar to computing $RT(F)$ as in Lemma SM6.1, except that
 889 now we do not require the origin to be fixed by the coordinate transformation $\Phi(x, y)$. How-
 890 ever, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ -symmetry forces $\Phi(0) = 0$, so $T(F) = RT(F)$.

891 We can now compute a universal unfolding of the normal form (5.24) using the analog of
 892 Theorem SM2.3.

893 *Proof of the Universal Unfolding Theorem 5.25:.* By Lemma SM6.4 and Remark SM6.3,
 894 a complement to $T(F)$ in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is

$$895 \quad \mathbb{R} \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\},$$

896 giving the universal unfolding (5.25).

897 References.

- 898 J. Damon. The unfolding and determinacy theorems for subgroups of \mathcal{A} and \mathcal{K} , *Mem. Am.*
 899 *Math. Soc.*, **50**, 306 (1984).
 900 J. Damon. Topological triviality and versality for subgroups \mathcal{A} and \mathcal{K} , *Mem. Am. Math. Soc.*
 901 **75**, 389 (1988).
 902 G. Dangelmayr and I. Stewart. Sequential bifurcations in continuous stirred tank reactors
 903 coupled in series, *SIAM J. Appl. Math.* **45**(6), 895–918 (1985)
 904 C. Gibson. *Singular Points of Smooth Mappings*, Research Notes in Mathematics **25**, Pitman,
 905 London 1979.
 906 M. Golubitsky and V. Guillemin. *Stable Mappings and Their Singularities*, Springer, New
 907 York 1973.
 908 M. Golubitsky and D.G. Schaeffer. *Singularities and Groups in Bifurcation Theory: Vol. I*,
 909 Appl. Math. Sci. **51**, Springer, New York 1985.
 910 M. Golubitsky and I. Stewart. Coordinate changes that preserve admissible maps for network
 911 dynamics, *Dynamical Systems* **32** (2017) 81–116.
 912 M. Golubitsky, I. Stewart, and D.G. Schaeffer. *Singularities and Groups in Bifurcation Theory:*
 913 *Vol. II*, Appl. Math. Sci. **69**, Springer, New York 1988.
 914 J. Martinet. *Singularities of Smooth Functions and Maps*, Cambridge University Press, Cam-
 915 bridge 1982.
 916 Wikipedia. Jacobi’s formula, https://en.wikipedia.org/wiki/Jacobi%27s_formula.