SUPPLEMENTARY MATERIALS: Shape versus Timing: Linear Responses of a Limit Cycle with Hard Boundaries under Instantaneous and Static Perturbation*

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In sections SM1–SM3 of this supplement, we prove Lemma 2.3 stated in section 2.2, derive equation (2.23) stated in section 2.3 and prove the main theorem 3.13. In section SM4, we present numerical algorithms for implementing all the methods reviewed and developed in section 2 and section 3. MATLAB code that implements these algorithms for the example system described in section 4 is available: https://github.com/yangyang-wang/LC_in_square. Section SM5 provides a table of symbols used in the main document.

Throughout, references to equations, theorems, lemmas and assumptions within the supplement begin with "SM", while references with just numbers refer to items in the main document.

SM1. Proof of Lemma 2.3. In this section we prove Lemma 2.3, which we restate for the reader's convenience.

Lemma SM1.1. *Let $\gamma_1^{\boldsymbol{a}}(t)$ and $\gamma_1^{\boldsymbol{b}}(t)$ be two T_0-periodic solutions to the iSRC equation (2.20) for a smooth vector field F_0 with a hyperbolically stable limit cycle $\gamma_0(t)$. Then, their difference satisfies $\gamma_1^{\boldsymbol{b}}(t) - \gamma_1^{\boldsymbol{a}}(t) = \varphi F_0(\gamma_0(t))$, where φ is a constant representing a fixed phase offset.*

Proof. Consider

$$\mathbf{x}' = F_0(\mathbf{x}) \tag{SM1.1}$$

with $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$. Let $\Phi(t, 0)$ be the fundamental matrix solution. Then $\Phi(t, 0)$ satisfies $\Phi'(t, 0) = DF_0(\mathbf{x}(t))\Phi(t, 0)$ and $\Phi(0, 0) = I$, where $\mathbf{x} = \gamma_0(t)$ is the unperturbed limit cycle solution. Suppose the monodromy matrix $M = \Phi(T_0, 0)$ is diagonalizable with eigenvalues $\{\mu_i, i = 1, \cdots, n\}$ associated with linearly independent eigenvectors $\{\mathbf{v}_i, i = 1, \cdots, n\}$. The eigenvalues μ_i are often referred to as Floquet multipliers of the periodic orbit solution $\gamma_0(t)$ of (SM1.1) (Meiss, 2007). Since $\gamma_0(t)$ is hyperbolically stable, M has a single trivial Floquet multiplier. Without loss of generality, we assume $\mu_1 = 1$ and hence $\mathbf{v}_1 = F_0(\gamma_0(0))$.

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Let the vector $\eta_i(t)$ be the solution to the variational equation

$$\eta_i' = DF_0(\gamma_0(t))\eta_i$$

that starts along the i-th Floquet eigenvector direction $\eta_i(0) = \mathbf{v}_i$. It follows that $\eta_i(T_0) = \Phi(T_0, 0)\eta_i(0) = M\mathbf{v}_i = \mu_i\mathbf{v}_i$. For simplicity, we denote $A(t) = DF_0(\gamma_0(t))$ hereafter.

Let $\rho_i = \ln(\mu_i)/T_0$ and let $\mathbf{q}_i(t) = e^{-t\rho_i}\eta_i(t)$ be the rescaled version of the trajectory $\eta_i(t)$, which is the i-th Floquet coordinate (Meiss, 2007). It follows from direct calculations that $\mathbf{q}_i(t)$ is periodic so that $\mathbf{q}_i(t) = \mathbf{q}_i(t + T_0)$ and satisfies the initial value problem

$$\mathbf{q}_i(t)' = A(t)\mathbf{q}_i(t) - \rho_i\mathbf{q}_i(t) \tag{SM1.2}$$

with $\mathbf{q}_i(0) = \mathbf{v}_i$. Since $\rho_1 = 0$, the first Floquet coordinate $\mathbf{q}_1(t)$ satisfies the initial value problem

$$\mathbf{q}_1' = A(t)\mathbf{q}_1, \quad \mathbf{q}_1(0) = F_0(\gamma_0(t)). \tag{SM1.3}$$

Using the chain rule, upon differentiating $F_0(\gamma_0(t))$, one can show that setting $\mathbf{q}_1(t) = F_0(\gamma_0(t))$ for any t, solves the initial value problem (SM1.3). Note that (SM1.2) is similar to the equation satisfied by $\mathcal{I}_i$, the gradient of isostable coordinates that are related to Floquet coordinates $\mathbf{q}_i$ (Wilson and Moehlis, 2016; Pérez-Cervera, Seara, and Huguet, 2020). In fact, direct calculation implies that the relationship between the iPRC and the variational dynamics (see Remark 2.2) also holds for $\mathbf{q}_i$ and $\mathcal{I}_i$; that is, $\frac{d(\mathbf{q}_i^\intercal \mathcal{I}_i)}{dt} = 0$ for $i = 1, \cdots, n$.

Note that at each t, the vector set $\{\mathbf{q}_i(t), i = 1, \cdots, n\}$ spans $\mathbb{R}^n$. Therefore we can write $\gamma_1(t)$, the general solution to the iSRC equation (2.20), as a linear combination of $\{\mathbf{q}_i(t), i = 1, \cdots, n\}$ with coefficients $a_i(t)$:

$$\gamma_1(t) = \sum_{i=1}^{n} a_i(t)\mathbf{q}_i(t).$$

Let $Q(t)$ be the $n \times n$ matrix $Q(t) = (\mathbf{q}_1(t)|\cdots|\mathbf{q}_n(t))$, and let $R = \text{diag}(\rho_1, \cdots, \rho_n)$ be the diagonal matrix with $\{\rho_1, \cdots, \rho_n\}$ as the diagonal entries, and $\mathbf{a}(t) = [a_1(t), \cdots, a_n(t)]^\intercal$. Then

$$\gamma_1(t) = Q(t)\mathbf{a}(t) \tag{SM1.4}$$

and (SM1.2) can be rewritten as

$$Q(t)' = A(t)Q(t) - Q(t)R. \tag{SM1.5}$$

Differentiating both sides of (SM1.4) and substituting in (SM1.5) leads to

$$\begin{aligned}
\gamma_1'(t) &= Q'(t)\mathbf{a}(t) + Q(t)\mathbf{a}'(t) \\
&= (A(t)Q(t) - Q(t)R)\mathbf{a}(t) + Q(t)\mathbf{a}'(t) \\
&= (Q(t)\mathbf{a}'(t) - Q(t)R\mathbf{a}(t)) + A(t)Q(t)\mathbf{a}(t) \\
&= (Q(t)\mathbf{a}'(t) - Q(t)R\mathbf{a}(t)) + A(t)\gamma_1(t).
\end{aligned}$$

On the other hand, by (2.20) we have

$$\gamma_1'(t) = A(t)\gamma_1(t) + \mathbf{c}(t),$$

where $\mathbf{c}(t) = \nu_1 F_0(\gamma_0(t)) + \frac{\partial F_\varepsilon(\gamma_0(t))}{\partial \varepsilon}\Big|_{\varepsilon=0}$. It follows that

$$Q(t)\mathbf{a}'(t) = Q(t)R\mathbf{a}(t) + \mathbf{c}(t). \tag{SM1.6}$$

Since $\{\mathbf{q}_i(t)\}_{i=1}^n$ spans $\mathbb{R}^b$ for each time $t \in [0, T_0]$, the matrix $Q(t)$ is invertible at each t. Thus multiplying both sides of (SM1.6) by $Q(t)^{-1}$ gives

$$\mathbf{a}'(t) = R\mathbf{a}(t) + Q(t)^{-1}\mathbf{c}(t). \tag{SM1.7}$$

Suppose two different iSRC curves are given by $\gamma_1^a(t) = \sum_{i=1}^n a_i(t)\mathbf{q}_i(t) = Q(t)\mathbf{a}(t)$ and $\gamma_1^b(t) = \sum_{i=1}^n b_i(t)\mathbf{q}_i(t) = Q(t)\mathbf{b}(t)$. Then

$$\mathbf{b}'(t) - \mathbf{a}'(t) = R\mathbf{b}(t) - R\mathbf{a}(t) = \begin{bmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_n \end{bmatrix} (\mathbf{b}(t) - \mathbf{a}(t)).$$

It follows that for $i = 1, \cdots, n$ we have $b_i(t) - a_i(t) = C_i e^{\rho_i t}$ for some constant C_i. Note that $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are both T_0-periodic. Therefore $C_i = C_i e^{\rho_i T_0}$. So either $C_i = 0$, and hence $a_i(t) \equiv b_i(t)$, or else $\rho_i = 0$. However, recall there is only one trivial multiplier $\mu_1 = 1$, so that only $\rho_1 = 0$ and $\rho_i \neq 0$ for $i = 2, \cdots, n$. Hence, $a_i(t) \equiv b_i(t)$ for $i = 2, \cdots, n$; that is, there exists some constant ϕ such that $\mathbf{b}(t) - \mathbf{a}(t) = \phi \mathbf{e}_1$.

Thus,

$$\gamma_1^b(t) - \gamma_1^a(t) = Q(t)\phi\mathbf{e}_1 = \phi\mathbf{q}_1(t).$$

Consequently, the two iSRC curves only differ along the first Floquet coordinate direction $\mathbf{q}_1(t) = F_0(\gamma_0(t))$ and hence only differ by a shift in phase along the direction of the limit cycle $\gamma_0(t)$:

$$\gamma_1^b(t) - \gamma_1^a(t) = \phi F_0(\gamma_0(t))$$

where ϕ is the constant phase shift introduced by the initial conditions $\gamma_1^b(0) - \gamma_1^a(0) = \phi F_0(\gamma_0(0))$. ■

SM2. Derivation of Equation (2.23).

This section establishes equation (2.23), which specifies the first-order change in the transit time through region I, or T_1^{I}:

$$T_1^{\mathrm{I}} = \eta^{\mathrm{I}}(\mathbf{x}^{\mathrm{in}}) \cdot \frac{\partial \mathbf{x}_\varepsilon^{\mathrm{in}}}{\partial \varepsilon}\Big|_{\varepsilon=0} + \int_{t^{\mathrm{in}}}^{t^{\mathrm{out}}} \eta^{\mathrm{I}}(\gamma(t)) \cdot \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon}\Big|_{\varepsilon=0} dt,$$

Recall $\mathcal{T}^{\mathrm{I}}(\mathbf{x})$ is the time remaining until exiting region I through Σ^{out}, under the unperturbed vector field, starting from location $\mathbf{x}$; $\eta^{\mathrm{I}} := \nabla\mathcal{T}^{\mathrm{I}}(\mathbf{x})$ is the local timing response curve (lTRC) for region I, defined for the component of the trajectory lying within region I, i.e. for times $t \in [t^{\mathrm{in}}, t^{\mathrm{out}}]$; and $\mathbf{x}_\varepsilon^{\mathrm{in}}$ is the coordinate of the perturbed entry point into region I.

We consider a single region $\mathcal{R}$ with entry surface Σ^{in} and exit surface Σ^{out}. We assume that these two surfaces are fixed, independent of static perturbation with size ε. The limit cycle solution $\mathbf{x} = \gamma_\varepsilon(\tau)$ satisfies

$$\frac{d\mathbf{x}}{d\tau} = F_\varepsilon(\mathbf{x})$$

where τ is the time coordinate of the perturbed trajectory. Moreover, $\gamma_\varepsilon(\tau)$ enters $\mathcal{R}$ at $\mathbf{x}_\varepsilon^{\text{in}} \in \Sigma^{\text{in}}$ when $\tau = t_\varepsilon^{\text{in}}$ and exits at $\mathbf{x}_\varepsilon^{\text{out}} \in \Sigma^{\text{out}}$ when $\tau = t_\varepsilon^{\text{out}}$. Since the system is autonomous, we are free to choose the reference time along the limit cycle orbit. For convenience of calculation, we set $t_\varepsilon^{\text{out}} \equiv 0$ for all ε.

Denote the transit time that γ_ε spends in $\mathcal{R}$ by $T_\varepsilon^{\mathcal{R}}$. It follows that $t_\varepsilon^{\text{in}} = -T_\varepsilon^{\mathcal{R}}$, where ε can be 0. Assuming that the transit time has a well behaved expansion in ε, we write

$$T_\varepsilon^{\mathcal{R}} = T_0^{\mathcal{R}} + \varepsilon T_1^{\mathcal{R}} + O(\varepsilon^2) \tag{SM2.1}$$

where $T_0^{\mathcal{R}}$ is the transit time for the unperturbed trajectory and $T_1^{\mathcal{R}}$ is the linear shift in the transit time. In the rest of this section, we drop the superscript $\mathcal{R}$ on $T_\varepsilon^{\mathcal{R}}$, $T_0^{\mathcal{R}}$ and $T_1^{\mathcal{R}}$ for simplicity.

Our goal is to prove that T_1 is given by (2.23). We do this in two steps. First, we show that the transit time T_ε can be expressed in terms of the perturbed vector field and perturbed local timing response curve (see (SM2.3)). Second, we expand the expression for T_ε to first order in ε to obtain the expression for T_1.

Since the time remaining to exit, denoted as $\mathcal{T}_\varepsilon$, decreases at a constant rate along trajectories, for arbitrary ε we have

$$-1 = \frac{d\mathcal{T}_\varepsilon}{d\tau} = F_\varepsilon(\gamma_\varepsilon(\tau)) \cdot \eta_\varepsilon(\gamma_\varepsilon(\tau)), \tag{SM2.2}$$

where $\eta_\varepsilon(\mathbf{x}) = \nabla \mathcal{T}_\varepsilon(\mathbf{x})$ is defined as the local timing response curve under perturbation. By (SM2.2), the transit time T_ε is therefore given by

$$T_\varepsilon = \int_{\tau = t_\varepsilon^{\text{out}}}^{t_\varepsilon^{\text{in}}} F_\varepsilon(\gamma_\varepsilon(\tau)) \cdot \eta_\varepsilon(\gamma_\varepsilon(\tau))\, d\tau. \tag{SM2.3}$$

In this expression, we integrate *backwards in time* along the limit cycle trajectory, from the egress point $\mathbf{x}_\varepsilon^{\text{out}}$ at time $t_\varepsilon^{\text{out}}$, to the ingress point $\mathbf{x}_\varepsilon^{\text{in}}$ at time $t_\varepsilon^{\text{in}}$:

For $\varepsilon = 0$, and taking into account (SM2.2), this integral reduces to

$$T_0 = \int_{\tau = t_0^{\text{out}}}^{t_0^{\text{in}}} F_0(\gamma_0(\tau)) \cdot \eta_0(\gamma_0(\tau))\, d\tau = \int_{\tau = t_0^{\text{out}}}^{t_0^{\text{in}}} (-1)\, d\tau = t_0^{\text{out}} - t_0^{\text{in}} = 0 - (-T_0), \tag{SM2.4}$$

since $t_0^{\text{in}} = -T_0$ and $t_\varepsilon^{\text{out}} \equiv 0$.

In order to derive an expression for T_1, the first order shift in the transit time, we need to expand (SM2.3) to first order in ε. To this end, we need to know the Taylor expansions for all terms in (SM2.3).

Suppose we can expand F_ε, $\mathcal{T}_\varepsilon$, and η_ε as follows:

$$\begin{aligned} F_\varepsilon(\mathbf{x}) &= F_0(\mathbf{x}) + \varepsilon F_1(\mathbf{x}) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0, \\ \mathcal{T}_\varepsilon(\mathbf{x}) &= \mathcal{T}_0(\mathbf{x}) + \varepsilon \mathcal{T}_1(\mathbf{x}) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0, \\ \eta_\varepsilon(\mathbf{x}) &= \eta_0(\mathbf{x}) + \varepsilon \eta_1(\mathbf{x}) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{SM2.5}$$

where $\eta_0(\mathbf{x}) = \nabla \mathcal{T}_0(\mathbf{x})$ is the unperturbed local timing response curve.

Following the idea of deriving the infinitesimal shape response curve in section 2.2, we write the portion of the perturbed limit cycle trajectory within region $\mathcal{R}$ in terms of the unperturbed limit cycle, plus a small correction,

$$\gamma_\varepsilon(\tau) = \gamma\left(\nu_\varepsilon \tau\right) + \varepsilon \gamma_1\left(\nu_\varepsilon \tau\right) + O(\varepsilon^2) \tag{SM2.6}$$

where $-T_\varepsilon \le \tau \le 0$ and $\nu_\varepsilon = \frac{T_0}{T_\varepsilon}$.

Now we expand (SM2.3) to first order

(SM2.7)

$$\begin{aligned} T_\varepsilon = &\int_{\tau=0}^{-T_\varepsilon} \Big[F_0(\gamma_0(\nu_\varepsilon\tau)) + \varepsilon DF_0(\gamma_0(\nu_\varepsilon\tau)) \cdot \gamma_1(\nu_\varepsilon\tau) + \varepsilon F_1(\gamma_0(\nu_\varepsilon\tau))\Big] \cdot \\ &\qquad \Big[\eta_0(\gamma_0(\nu_\varepsilon\tau)) + \varepsilon D\eta_0(\gamma_0(\nu_\varepsilon\tau)) \cdot \gamma_1(\nu_\varepsilon\tau) + \varepsilon \eta_1(\gamma_0(\nu_\varepsilon\tau))\Big] d\tau + O(\varepsilon^2) \\ = &\int_{\tau=0}^{-T_\varepsilon} F_0(\gamma_0(\nu_\varepsilon\tau)) \cdot \eta_0(\gamma_0(\nu_\varepsilon\tau)) d\tau + \varepsilon \Big[F_0(\gamma_0(\nu_\varepsilon\tau)) \cdot \eta_1(\gamma_0(\nu_\varepsilon\tau)) + F_1(\gamma_0(\nu_\varepsilon\tau)) \cdot \eta_0(\gamma_0(\nu_\varepsilon\tau))\Big] d\tau + \\ &\qquad \varepsilon \Big[F_0(\gamma_0(\nu_\varepsilon\tau)) \cdot D\eta_0(\gamma_0(\nu_\varepsilon\tau)) \cdot \gamma_1(\nu_\varepsilon\tau) + DF_0(\gamma_0(\nu_\varepsilon\tau)) \cdot \gamma_1(\nu_\varepsilon\tau) \cdot \eta_0(\gamma_0(\nu_\varepsilon\tau))\Big] d\tau + O(\varepsilon^2) \\ = &\frac{1}{\nu_\varepsilon} \int_{t=0}^{-T_0} F_0(\gamma_0(t)) \cdot \eta_0(\gamma_0(t)) dt + \varepsilon \Big[F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))\Big] dt + \\ &\qquad \varepsilon \Big[F_0(\gamma_0(t)) \cdot D\eta_0(\gamma_0(t)) \cdot \gamma_1(t) + DF_0(\gamma_0(t)) \cdot \gamma_1(t) \cdot \eta_0(\gamma_0(t))\Big] dt + O(\varepsilon^2) \end{aligned}$$

To order $O(1)$, we recover

$$T_0 = \int_{t=0}^{-T_0} F_0(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))\, dt. \tag{SM2.8}$$

This leads to $T_\varepsilon = \frac{1}{\nu_\varepsilon} T_0$, as required for consistency. We are therefore left with

$$\begin{aligned} 0 = &\int_{t=0}^{-T_0} \Big[F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))\Big] dt \\ &+ \int_{t=0}^{-T_0} \Big[F_0(\gamma_0(t)) \cdot D\eta_0(\gamma_0(t)) \cdot \gamma_1(t) + DF_0(\gamma_0(t)) \cdot \gamma_1(t) \cdot \eta_0(\gamma_0(t))\Big] dt + O(\varepsilon) \\ = &\int_{t=0}^{-T_0} \Big[F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))\Big] dt \\ &+ \int_{t=0}^{-T_0} \Big[F_0(\gamma_0(t))^\intercal D\eta_0(\gamma_0(t)) + \eta_0(\gamma_0(t))^\intercal DF_0(\gamma_0(t))\Big] \cdot \gamma_1(t) dt + O(\varepsilon) \end{aligned} \tag{SM2.9}$$

where the second equality follows from rearranging orders of factors in the second integral.

Note that since $F_0 \cdot \eta_0 \equiv -1$ everywhere, we have the identity

$$0 = \frac{\partial}{\partial \mathbf{x}_j}\left(\sum_i \eta^i F^i\right) = \sum_i \frac{\partial \eta^i}{\partial \mathbf{x}_j} F^i + \sum_i \eta^i \frac{\partial F^i}{\partial \mathbf{x}_j} \tag{SM2.10}$$

where F^i and η^i are the i-th components for F_0 and η_0; $\mathbf{x}_j$ denotes the jth component of $\mathbf{x}$ for $j \in \{1, \cdots, n\}$. It follows that $F_0^\intercal(D\eta_0) + \eta_0^\intercal(DF_0) = 0$ in (SM2.9), leaving only

$$0 = \int_{t=0}^{-T_0} \Big[F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t)) + F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))\Big]\, dt. \tag{SM2.11}$$

Since $F_0(\gamma_0(t)) = d\gamma_0/dt$ and $\eta_1(\mathbf{x}) = \partial\eta_\varepsilon(\mathbf{x})/\partial\varepsilon|_{\varepsilon=0} = \partial\nabla\mathcal{T}_\varepsilon(\mathbf{x})/\partial\varepsilon|_{\varepsilon=0}$,

$$\begin{aligned}
\int_{t=0}^{-T_0} F_0(\gamma_0(t)) \cdot \eta_1(\gamma_0(t))\, dt &= \int_{t=0}^{-T_0} \left(\frac{d\gamma_0}{dt}\right) \cdot \left(\frac{\partial}{\partial\varepsilon}\left[\nabla\mathcal{T}_\varepsilon(\gamma_0(t))\right]\right)\Bigg|_{\varepsilon=0} dt \\
&= \int_{t=0}^{-T_0} \left(\frac{d\gamma_0}{dt}\right) \cdot \nabla\left(\frac{\partial}{\partial\varepsilon}\left[\mathcal{T}_\varepsilon(\gamma_0(t))\right]\right)\Bigg|_{\varepsilon=0} dt \\
&= \int_{t=0}^{-T_0} \frac{d}{dt}\left(\frac{\partial}{\partial\varepsilon}\left[\mathcal{T}_\varepsilon(\gamma_0(t))\right]\right)\Bigg|_{\varepsilon=0} dt \\
&= \left(\frac{\partial}{\partial\varepsilon}\left[\mathcal{T}_\varepsilon(\mathbf{x}_0^{\text{in}})\right]\right)\Bigg|_{\varepsilon=0} - \left(\frac{\partial}{\partial\varepsilon}\left[\mathcal{T}_\varepsilon(\mathbf{x}_0^{\text{out}})\right]\right)\Bigg|_{\varepsilon=0} \\
&= \left(\frac{\partial}{\partial\varepsilon}\left[\mathcal{T}_\varepsilon(\mathbf{x}_0^{\text{in}})\right]\right)\Bigg|_{\varepsilon=0} - 0 \\
&= \mathcal{T}_1(\mathbf{x}_0^{\text{in}}).
\end{aligned}$$

Therefore

$$\mathcal{T}_1(\mathbf{x}_0^{\text{in}}) = \int_{t=-T_0}^{0} F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))dt = \int_{t=t_0^{\text{in}}}^{t_0^{\text{out}}} F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))dt. \tag{SM2.12}$$

The second equality follows from our convention that $t_0^{\text{in}} = -T_0$ and $t_\varepsilon^{\text{out}} \equiv 0$.

We notice that

$$T_\varepsilon = \mathcal{T}_\varepsilon(\mathbf{x}_\varepsilon^{\text{in}}) = T_0 + \varepsilon\left(\mathcal{T}_1(\mathbf{x}_0^{\text{in}}) + \nabla\mathcal{T}_0(\mathbf{x}_0^{\text{in}}) \cdot \mathbf{x}_1^{\text{in}}\right), \tag{SM2.13}$$

where we have made use of the Taylor expansion $\mathbf{x}_\varepsilon^{\text{in}} = \mathbf{x}_0^{\text{in}} + \varepsilon\mathbf{x}_1^{\text{in}} + O(\varepsilon^2)$, as $\varepsilon \to 0$. Equating the first order terms in (SM2.1) and (SM2.13) leads to

$$T_1 = \mathcal{T}_1(\mathbf{x}_0^{\text{in}}) + \eta_0(\mathbf{x}_0^{\text{in}}) \cdot \mathbf{x}_1^{\text{in}}. \tag{SM2.14}$$

Substituting (SM2.12) into (SM2.14), we finally obtain

$$T_1 = \eta_0(\mathbf{x}_0^{\text{in}}) \cdot \mathbf{x}_1^{\text{in}} + \int_{t=t_0^{\text{in}}}^{t_0^{\text{out}}} F_1(\gamma_0(t)) \cdot \eta_0(\gamma_0(t))dt \tag{SM2.15}$$

which is (2.23), as desired.

SM3. Proof of Theorem 3.13. In this section we present a proof of Theorem 3.13, which we restate for the reader's convenience. As discussed before, parts (a) and (b) are already covered in (Filippov, 1988; Bernardo et al., 2008), whereas parts (c) through (d) are our new results. For completeness, we still include parts (a) and (b) as well as our versions of proofs.

Theorem SM3.1. *Consider a general LCSC described locally by (3.7) in the neighborhood of a hard boundary Σ, satisfying Assumption 3.10 and Assumption 3.12. The following properties hold for the variational dynamics $\mathbf{u}$ and the iPRC $\mathbf{z}$ along Σ:*
- (a) *At the landing point of Σ, the saltation matrix is $S = I - nn^\intercal$, where I is the identity matrix.*
- (b) *At the liftoff point of Σ, the saltation matrix is $S = I$.*
- (c) *Along the sliding region within Σ, the component of $\mathbf{z}$ normal to Σ is zero.*
- (d) *The normal component of $\mathbf{z}$ is continuous at the landing point.*
- (e) *The tangential components of $\mathbf{z}$ are continuous at both landing and liftoff points.*

Proof. We choose coordinates $\mathbf{x} = (\mathbf{w}, v) = (w_1, w_2, \ldots, w_{n-1}, v)$ so that within a neighborhood containing both the landing and liftoff points, the hard boundary corresponds to $v = 0$, the interior of the domain coincides with $v > 0$, and the unit normal vector for the hard boundary is $\mathbf{n} = (0, \ldots, 0, 1)$. In these coordinates, the velocity vector is $\mathbf{F} = (f_1, f_2, \ldots, f_{n-1}, g)$. In addition, we use $\mathbf{F}^{\text{slide}}$ to denote the vector field for points on the sliding region, whereas the dynamics of other points is governed by $\mathbf{F}^{\text{int}}$. The transversal intersection condition for the trajectory entering the hard boundary is $g^{\text{int}}(\mathbf{x}_{\text{land}}, 0) < 0$ (cf. eq. (3.3); note that $\mathbf{n}$ defined here points in the opposite direction from the *outward* normal vector in (3.3)). At points $\mathbf{x} \in \mathcal{L}$ on the liftoff boundary, $\mathbf{F}^{\text{slide}}$ and $\mathbf{F}^{\text{int}}$ coincide and we will use whichever notation seems clearer in a given instance. Under the nondegeneracy condition at the liftoff point (3.6), we can further arrange the coordinates $(w_1, \ldots, w_{n-1})$ so that the unit vector normal to the liftoff boundary $\mathcal{L}$ at the liftoff point is $\ell = (0, \ldots, 0, 1, 0)$, and $g^{\text{int}} \gtrless 0 \iff w_{n-1} \gtrless 0$. With these coordinates, the nondegeneracy condition (3.6) is $\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}}) \cdot \ell = f^{\text{slide}}_{n-1}(\mathbf{x}_{\text{lift}}) > 0$.

(a) *At the landing point, the saltation matrix is $S = I - \mathbf{n}\mathbf{n}^\intercal$, where I is the identity matrix.* The saltation matrix at a transition from the interior to a sliding motion along a hard boundary is given in (Bernardo et al. (2008), Example 2.14, p. 111) as

$$S = I + \frac{(\mathbf{F}^{\text{slide}} - \mathbf{F}^{\text{int}})\mathbf{n}^\intercal}{\mathbf{n}^\intercal \mathbf{F}^{\text{int}}}, \tag{SM3.1}$$

provided the trajectory approaches the hard boundary transversally.

It follows from the definition of the sliding vector field $\mathbf{F}^{\text{slide}}$ given by (3.4) that

$$S = I - \mathbf{n}\mathbf{n}^\intercal,$$

as claimed.

(b) *At the liftoff point, the saltation matrix is $S = I$.* We adapt the argument in (Bernardo et al. (2008), §2.5) to our hard boundary/liftoff construction. The essential difference is that the trajectory is not transverse to the hard boundary at the liftoff point, indeed $\mathbf{n}^\intercal \mathbf{F} = 0$ at $\mathbf{x}_{\text{lift}}$, so eq. (SM3.1) does not give a well defined saltation matrix. However, by replacing the vector $\mathbf{n}$ normal to the hard boundary with the vector ℓ normal to the liftoff boundary, we recover an equation analogous to (SM3.1), as we will show. Since $\mathbf{F}^{\text{slide}} = \mathbf{F}^{\text{int}}$ at the liftoff point, we conclude that the saltation matrix at the liftoff point reduces to the identity matrix.

Let Φ_{I} and Φ_{II} denote the flow operators on the sliding region and in the domain complementary to the sliding region, respectively. That is, $\Phi_{\text{I}}(\mathbf{x}, t)$ takes initial point $\mathbf{x} \in \mathcal{R}^{\text{slide}}$ at time zero to $\Phi_{\text{I}}(\mathbf{x}, t)$ at time $0 \le t \le \mathcal{T}(\mathbf{x})$. So Φ_{I} is restricted to act for times up to the time $\mathcal{T}(\mathbf{x})$ at which the trajectory starting at $\mathbf{x}$ reaches the liftoff point, $\Phi_{\text{I}}(\mathbf{x}, \mathcal{T}(\mathbf{x})) \in \mathcal{L}$. Such a trajectory necessarily has initial condition $\mathbf{x} = (w_1, \ldots, w_{n-1}, 0)$ satisfying $w_{n-1} < 0$, by our coordinatization. Let $\mathbf{x}_{\text{a}} \in \mathcal{R}^{\text{slide}}$ be a point on the periodic limit cycle solution, so that $\Phi_{\text{I}}(\mathbf{x}_{\text{a}}, \mathcal{T}(\mathbf{x}_{\text{a}})) = \mathbf{x}_{\text{lift}}$. Write $\tau = \mathcal{T}(\mathbf{x}_{\text{a}})$ for the time it takes for the trajectory to reach the liftoff point after passing location $\mathbf{x}_{\text{a}}$. We require a first-order accurate estimate of the effect of the boundary on the displacement between the unperturbed trajectory and a nearby trajectory. If we make a small (size ε) perturbation into the domain interior, away from the constraint surface, the normal component of the perturbed trajectory will return to zero within a time interval of $O(\varepsilon)$ duration, before the two trajectories reach the liftoff boundary. Therefore we need only consider perturbations tangent to the constraint surface.

Let $\mathbf{x}'_{\text{a}} \in \mathcal{R}^{\text{slide}}$ denote a point near $\mathbf{x}_{\text{a}}$, and suppose it takes time $\mathcal{T}(\mathbf{x}'_{\text{a}}) = \tau + \delta$ for the trajectory through $\mathbf{x}'_{\text{a}}$ to lift off, at some point $\mathbf{x}'_{\text{lift}} \in \mathcal{L}$. There are two cases to consider: either $\delta \ge 0$ or else $\delta \le 0$. The two cases are handled similarly; we focus on the first for brevity. In case $\delta > 0$, the original trajectory arrives at the liftoff boundary before the perturbed trajectory, and the point $\mathbf{x}'_{\text{b}} = \Phi_{\text{I}}(\mathbf{x}'_{\text{a}}, \tau) \in \mathcal{R}^{\text{slide}}$. We write $\mathbf{x}'_{\text{b}} = \mathbf{x}_{\text{lift}} + \Delta\mathbf{x}_{\text{b}}$ (see Fig. SM1B) and expand the flow operator as follows:

$$\begin{aligned}\Phi_{\text{I}}(\mathbf{x}'_{\text{b}}, \delta) =& \mathbf{x}'_{\text{b}} + \delta\, \mathbf{F}^{\text{slide}}(\mathbf{x}'_{\text{b}}) + \frac{\delta^2}{2}\left(\nabla^{\text{slide}}\mathbf{F}^{\text{slide}}(\mathbf{x}'_{\text{b}})\right)\cdot \mathbf{F}^{\text{slide}}(\mathbf{x}'_{\text{b}}) + O(\delta^3)\\ =& \mathbf{x}_{\text{lift}} + \Delta\mathbf{x}_{\text{b}} + \delta\, \mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}}) + \delta\left(\nabla^{\text{slide}}\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})\right)\cdot\Delta\mathbf{x}_{\text{b}}\\ &+ \frac{\delta^2}{2}\left(\nabla^{\text{slide}}\mathbf{F}^{\text{slide}}(\mathbf{x}'_{\text{b}})\right)\cdot\mathbf{F}^{\text{slide}}(\mathbf{x}'_{\text{b}}) + O(3),\end{aligned} \tag{SM3.2}$$

where ∇^{slide} is the gradient operator restricted to $\mathbf{x} = (x_1, \ldots, x_{n-1})$. The Taylor expansion in (SM3.2) is justified in a neighborhood of $\mathbf{x}'_{\text{b}}$ contained in the sliding region of the hard boundary. The transversality of the intersection of the reference trajectory with $\mathcal{L}$ (that is, $\mathbf{F}_{n-1}(\mathbf{x}_{\text{lift}}) > 0$) means that δ and $|\Delta\mathbf{x}_{\text{b}}|$ will be of the same order. We write $O(n)$ to denote terms of order $(|\Delta\mathbf{x}_{\text{b}}|^p\delta^{n-p})$ for $0 \le p \le n$.

Next we estimate δ and the location $\mathbf{x}'_{\text{lift}}$ at which the perturbed trajectory crosses $\mathcal{L}$. To first order,

$$\ell^\intercal \mathbf{x}'_{\text{b}} = \ell^\intercal \mathbf{F}^{\text{slide}}(\mathbf{x}'_{\text{b}})\,\delta \tag{SM3.3}$$

$$\ell^\intercal(\mathbf{x}_{\text{lift}} + \Delta\mathbf{x}_{\text{b}}) = \ell^\intercal\left(\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}} + \Delta\mathbf{x}_{\text{b}})\right)\delta \tag{SM3.4}$$

$$\begin{aligned}\ell^\intercal\Delta\mathbf{x}_{\text{b}} &= \ell^\intercal\left(\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}}) + \left(\nabla^{\text{slide}}\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})\right)\cdot\Delta\mathbf{x}_{\text{b}}\right)\delta\\ &= \ell^\intercal\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})\delta + O(2)\end{aligned} \tag{SM3.5}$$

$$\delta = \frac{\ell^\intercal\Delta\mathbf{x}_{\text{b}}}{\ell^\intercal\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})} + O(2). \tag{SM3.6}$$

Combining this result with (SM3.2), the perturbed trajectory's liftoff location is

$$\mathbf{x}'_{\text{lift}} = \mathbf{x}_{\text{lift}} + \Delta\mathbf{x}_{\text{b}} + \mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})\delta + O(2). \tag{SM3.7}$$

Meanwhile, as the perturbed trajectory proceeds to $\mathcal{L}$, during a time interval of duration δ, the unperturbed trajectory has reentered the interior and evolves according to Φ_{II}, the flow defined for all initial conditions *not* within the sliding region. At a time δ after reaching $\mathcal{L}$, the unperturbed trajectory is located, to first order, at a point

$$\mathbf{x}_{\text{c}} = \mathbf{x}_{\text{lift}} + \mathbf{F}^{\text{int}}(\mathbf{x}_{\text{lift}})\,\delta + O(2). \tag{SM3.8}$$

Thus, combining (SM3.7) and (SM3.8) the displacement between the two trajectories immediately following liftoff of the perturbed trajectory, $\Delta\mathbf{x}_{\text{c}} = \mathbf{x}'_{\text{lift}} - \mathbf{x}_{\text{c}}$, is given (to first order) by

$$\begin{aligned}
\Delta\mathbf{x}_{\text{c}} &= \mathbf{x}'_{\text{lift}} - \mathbf{x}_{\text{c}} \\
&= \mathbf{x}_{\text{lift}} + \Delta\mathbf{x}_{\text{b}} + \mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})\delta - \left(\mathbf{x}_{\text{lift}} + \mathbf{F}^{\text{int}}(\mathbf{x}_{\text{lift}})\,\delta\right) \\
&= \Delta\mathbf{x}_{\text{b}} + \left(\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}}) - \mathbf{F}^{\text{int}}(\mathbf{x}_{\text{lift}})\right)\delta \\
&= \Delta\mathbf{x}_{\text{b}} + \frac{\left(\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}}) - \mathbf{F}^{\text{int}}(\mathbf{x}_{\text{lift}})\right)\ell^{\intercal}\Delta\mathbf{x}_{\text{b}}}{\ell^{\intercal}\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})} \\
&= S_{\text{lift}}\Delta\mathbf{x}_{\text{b}} + O(2).
\end{aligned}$$

Therefore, the saltation matrix at the liftoff point is

$$S_{\text{lift}} = I + \frac{\left(\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}}) - \mathbf{F}^{\text{int}}(\mathbf{x}_{\text{lift}})\right)\ell^{\intercal}}{\ell^{\intercal}\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}})}. \tag{SM3.9}$$

We take the vector field on the sliding region to be the projection of the vector field defined for the interior onto the boundary surface (cf. (3.4)). Therefore for our construction $\mathbf{F}^{\text{slide}}(\mathbf{x}_{\text{lift}}) = \mathbf{F}^{\text{int}}(\mathbf{x}_{\text{lift}})$, and hence $S_{\text{lift}} = I$, as claimed. We note that equation (SM3.9) will hold for more general constructions as well. This concludes the proof of part (b).

In parts (c) and (d) of the proof, our goal is to show the normal component of the iPRC is zero along the sliding region on Σ and is continuous at the landing point. To this end, we compute the normal component of the iPRC using its definition (2.8), which in $(\mathbf{w}, v)$ coordinates takes the form

$$\mathbf{z}_v := \mathbf{z}\cdot\mathbf{n} = \lim_{\varepsilon\to 0}\frac{\phi(\mathbf{x}+\varepsilon\mathbf{n}) - \phi(\mathbf{x})}{\varepsilon}, \tag{SM3.10}$$

where $\phi(\mathbf{x})$ denotes the asymptotic phase at point $\mathbf{x}$ on the limit cycle. That is, we apply a small instantaneous perturbation to the limit cycle, either while it is sliding along Σ (part c) or else just before landing (part d), in the $\mathbf{n}$ direction, and estimate the phase difference between the perturbed and unperturbed trajectories (cf. Fig. SM1).

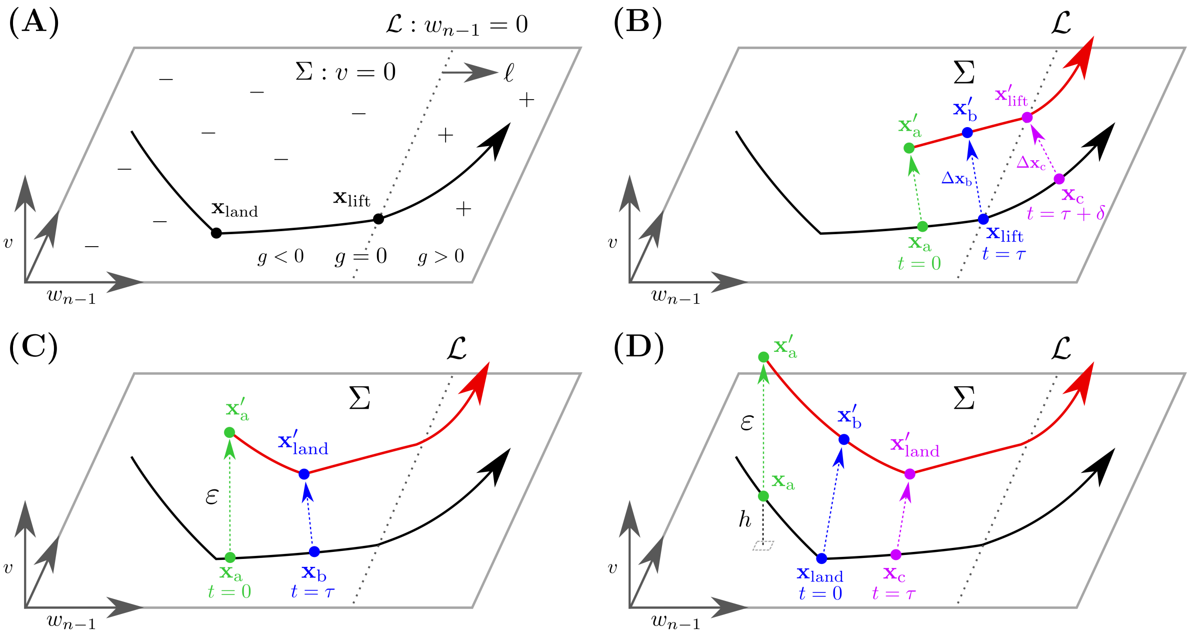

Figure SM1. *Unperturbed trajectory (black curve) and a perturbed trajectory (red curve) near the hard boundary* Σ *(horizontal plane) in the* $(\mathbf{w}, v)$ *phase space. Dashed line: intersection of liftoff boundary* $\mathcal{L}$ *and* Σ. (A) *Trajectory moves downward towards the sliding region (the area in* Σ *where* $g < 0$*), hits* Σ *at the landing point* $\mathbf{x}_{\mathrm{land}}$*, and exits* Σ *at the liftoff point* $\mathbf{x}_{\mathrm{lift}}$*.* (B) *Construction for the proof of part* (b)*. An instantaneous perturbation tangent to* Σ *is made to the point* $\mathbf{x}_a$ *at* $t = 0$*, pushing it to a point* $\mathbf{x}'_a \in \Sigma$*. The trajectory starting at* $\mathbf{x}_a$ *(resp.,* $\mathbf{x}'_a$*) reaches the liftoff point* $\mathbf{x}_{\mathrm{lift}}$ *(resp.,* $\mathbf{x}'_b$*) after time* τ*, and reaches* $\mathbf{x}_c$ *(resp.,* $\mathbf{x}'_{\mathrm{lift}}$*) after additional time* δ*. The displacements* $\Delta\mathbf{x}_b = \mathbf{x}'_b - \mathbf{x}_{\mathrm{lift}}$ *and* $\Delta\mathbf{x}_c = \mathbf{x}'_{\mathrm{lift}} - \mathbf{x}_c$ *differ by an amount captured, to linear order, by the saltation matrix.* (C) *Construction for the proof of part* (c)*. An instantaneous perturbation with size* ε *in the positive* v*-direction (green arrow) is made to the point* $\mathbf{x}_a \in \Sigma$*, pushing it off the boundary to an interior point* $\mathbf{x}'_a$*. After time* τ*, the trajectory starting at* $\mathbf{x}'_a$ *(resp.,* $\mathbf{x}_a$*) reaches a landing point* $\mathbf{x}'_{\mathrm{land}}$ *(resp.,* $\mathbf{x}_b$*).* (D) *The same perturbation (green arrow) as in panel* (C) *is applied to the point* $\mathbf{x}_a$ *located at a distance of* h *above* Σ*, pushing it to a point* $\mathbf{x}'_a$*. The trajectory starting at* $\mathbf{x}_a$ *lands on* Σ *at* $\mathbf{x}_{\mathrm{land}}$*. After the same amount of time, the perturbed trajectory starting at* $\mathbf{x}'_a$ *reaches* $\mathbf{x}'_b$*. After additional time* τ*, the two trajectories reach* $\mathbf{x}_c$ *and* $\mathbf{x}'_{\mathrm{land}}$*, respectively.*

(c) *Along the sliding region, the component of* $\mathbf{z}$ *normal to* Σ *is zero.* By (SM3.10) the normal component of the iPRC for a point on the sliding component of the trajectory, denoted by $\mathbf{x}_{\mathrm{a}} = (w_{\mathrm{a}}, 0)$ is given by

$$\mathbf{z}_v(\mathbf{x}_{\mathrm{a}}) = \lim_{\varepsilon \to 0} \frac{\phi(w_{\mathrm{a}}, \varepsilon) - \phi(w_{\mathrm{a}}, 0)}{\varepsilon}. \tag{SM3.11}$$

By $\mathbf{x}'_{\mathrm{a}} = (w_{\mathrm{a}}, \varepsilon)$ we denote a point that is located at a distance of ε above $\mathbf{x}_{\mathrm{a}}$. Our goal is to show $\mathbf{z}_v(\mathbf{x}_{\mathrm{a}}) = 0$.

The perturbed trajectory from $\mathbf{x}'_{\mathrm{a}}$ is governed by the interior flow Φ_{II} until it reaches the sliding region at a point $\mathbf{x}'_{\mathrm{land}} \in \Sigma$, after some time τ. Meanwhile the unperturbed trajectory from $\mathbf{x}_{\mathrm{a}}$ is governed by the sliding flow Φ_{I} until it crosses the liftoff point at $\mathcal{L}$ (Fig. SM1, dotted line).

To first order in ε, the time for the perturbed trajectory $\mathbf{x}'(t)$ to return to the constraint surface is

$$\begin{aligned}\tau(\varepsilon) &= -\frac{\varepsilon}{g^{\text{int}}(\mathbf{w}_\text{a},\varepsilon)} + O(\varepsilon^2) = -\frac{\varepsilon}{g^{\text{int}}(\mathbf{w}_\text{a},0)+\varepsilon D_v g^{\text{int}}(\mathbf{w}_\text{a},0)+O(\varepsilon^2)} + O(\varepsilon^2)\\ &= -\frac{\varepsilon}{g^{\text{int}}(\mathbf{w}_\text{a},0)} + O(\varepsilon^2), \text{ as } \varepsilon\to 0.\end{aligned} \tag{SM3.12}$$

Because $\mathbf{x}_\text{a} = (\mathbf{w}_\text{a}, 0)$ is in the sliding region, $g^{\text{int}}(\mathbf{w}_\text{a}, 0) < 0$; we conclude that τ and ε are of the same order. We use (p) to denote terms of order p in ε or τ.

At time τ following the perturbation, the location of the perturbed trajectory is

$$\begin{aligned}\mathbf{x}'_\text{land} &= \Phi_{\text{II}}(\mathbf{x}'_\text{a},\tau)\\ &= \mathbf{x}'_\text{a} + \tau\mathbf{F}^{\text{int}}(\mathbf{x}'_\text{a}) + O(2)\\ &= \mathbf{x}_\text{a} + \varepsilon\mathbf{n} + \tau\left(\mathbf{F}^{\text{int}}(\mathbf{x}_\text{a}) + \varepsilon\mathbf{n}^\intercal D\mathbf{F}^{\text{int}}(\mathbf{x}_\text{a})\right) + O(2)\\ &= \mathbf{x}_\text{a} + (0,\dots,0,\varepsilon) - \frac{\varepsilon}{g^{\text{int}}(\mathbf{x}_\text{a})}(f_1^{\text{int}}(\mathbf{x}_\text{a}),\dots,f_{n-1}^{\text{int}}(\mathbf{x}_\text{a}), g^{\text{int}}(\mathbf{x}_\text{a})) + O(2)\\ &= \mathbf{x}_\text{a} - \frac{\varepsilon}{g^{\text{int}}(\mathbf{x}_\text{a})}(f_1^{\text{int}}(\mathbf{x}_\text{a}),\dots,f_{n-1}^{\text{int}}(\mathbf{x}_\text{a}),0) + O(2).\end{aligned} \tag{SM3.13}$$

Simultaneously, the location of the unperturbed trajectory is

$$\begin{aligned}\mathbf{x}_\text{b} &= \Phi_{\text{I}}(\mathbf{x}_\text{a},\tau)\\ &= \mathbf{x}_\text{a} + \tau\mathbf{F}^{\text{slide}}(\mathbf{x}_\text{a}) + O(2)\\ &= \mathbf{x}_\text{a} - \frac{\varepsilon}{g^{\text{int}}(\mathbf{x}_\text{a})}(f_1^{\text{int}}(\mathbf{x}_\text{a}),\dots,f_{n-1}^{\text{int}}(\mathbf{x}_\text{a}),0) + O(2),\end{aligned} \tag{SM3.14}$$

since for $\mathbf{x} \in \Sigma$, we have $f^{\text{slide}}(\mathbf{x}) = f^{\text{int}}(\mathbf{x})$ by construction. Comparing the difference in location of the two trajectories at time τ after the perturbation, we see that

$$||\mathbf{x}'_\text{land} - \mathbf{x}_b|| = O(\varepsilon^2). \tag{SM3.15}$$

By assumption, the asymptotic phase function $\phi(\mathbf{x})$ is C^1 with respect to displacements tangent to the constraint surface. Since both $\mathbf{x}_\text{b}$ and $\mathbf{x}'_\text{land}$ are on this surface, $\mathbf{n}^\intercal(\mathbf{x}'_\text{land} - \mathbf{x}_\text{b}) = 0$, and $\phi(\mathbf{x}'_\text{land}) = \phi(\mathbf{x}_\text{b}) + O(\varepsilon^2)$. Therefore $\mathbf{z}_v(\mathbf{x}_\text{a}) = 0$ for points $\mathbf{x}_\text{a}$ on the sliding component of the limit cycle. This completes the proof of part (c).

(d) *The normal component of* $\mathbf{z}$ *is continuous at the landing point.* In order to show that the normal component of the iPRC ($\mathbf{z}_v$) is continuous at the landing point, we prove that $\mathbf{z}_v$ has a well-defined limit at the landing point and moreover, this limit equals 0 which is the value of $\mathbf{z}_v$ at the landing point as proved in (c). To this end, consider a point on the limit cycle shortly ahead of the landing point, $\mathbf{x}_\text{a} = (\mathbf{w}_\text{a}, h)$ with $0 < h \ll 1$ fixed, (cf. Fig. SM1D). By (SM3.10)

$$\mathbf{z}_v(\mathbf{x}_a) = \lim_{\varepsilon\to 0}\tfrac{\phi(\mathbf{w}_\text{a},h+\varepsilon)-\phi(\mathbf{w}_\text{a},h)}{\varepsilon}. \tag{SM3.16}$$

Our goal is to show $\lim_{h\to 0}\mathbf{z}_v(\mathbf{x}_a) = \mathbf{z}_v(\mathbf{x}_\text{land}) = 0$.

We consider the case $\varepsilon > 0$; the treatment for $\varepsilon < 0$ is similar. For $\varepsilon > 0$, when the unperturbed trajectory arrives at the constraint surface (at landing point $\mathbf{x}_{\text{land}}$), the perturbed trajectory is at a point $\mathbf{x}'_{\text{b}}$ that is still in the interior of the domain. Denote the unperturbed landing time $t = 0$; denote the time of flight from initial point $\mathbf{x}_{\text{a}}$ to $\mathbf{x}_{\text{land}}$ by s. Through an estimate similar to that in part (c), to first order in h, we have

$$s(h) = -\frac{h}{g^{\text{int}}(\mathbf{x}_{\text{land}})} + O(h^2). \tag{SM3.17}$$

Between $t = -s$ and $t = 0$, the displacement between the perturbed trajectory ($\mathbf{x}'(t)$) and the unperturbed trajectory ($\mathbf{x}(t)$) satisfies

$$\frac{d(\mathbf{x}' - \mathbf{x})}{dt} = D\mathbf{F}^{\text{int}}(\mathbf{x}(t)) \cdot (\mathbf{x}' - \mathbf{x}) + O(\varepsilon^2), \tag{SM3.18}$$

with initial condition $\mathbf{x}'(-s) - \mathbf{x}(-s) = \varepsilon\mathbf{n}$. Because the interior vector field is presumed C^1, for $h, s \ll 1$ we have

$$\begin{aligned}
\mathbf{x}'_{\text{b}} - \mathbf{x}_{\text{land}} &= \mathbf{x}'_{\text{a}} - \mathbf{x}_{\text{a}} + s\left(\varepsilon D_v \mathbf{F}^{\text{int}}(\mathbf{x}_{\text{land}}) + O(\varepsilon^2)\right) + O(s^2) \\
&= \varepsilon\mathbf{n} - h\left(\varepsilon\frac{D_v\mathbf{F}^{\text{int}}(\mathbf{x}_{\text{land}})}{g^{\text{int}}(\mathbf{x}_{\text{land}})} + O(\varepsilon^2)\right) + O(h^2) \\
&= (0, \cdots, 0, \varepsilon) - \frac{h\varepsilon}{g^{\text{int}}(\mathbf{x}_{\text{land}})}(f^{\text{int}}_{1,v}(\mathbf{x}_{\text{land}}), \ldots, f^{\text{int}}_{n-1,v}(\mathbf{x}_{\text{land}}), g^{\text{int}}_v(\mathbf{x}_{\text{land}})) + O(2) \\
&= \left(-h\varepsilon\frac{\mathbf{f}^{\text{int}}_v(\mathbf{x}_{\text{land}})}{g^{\text{int}}(\mathbf{x}_{\text{land}})}, \varepsilon - h\varepsilon\frac{g^{\text{int}}_v(\mathbf{x}_{\text{land}})}{g^{\text{int}}(\mathbf{x}_{\text{land}})}\right) + O(2).
\end{aligned}$$

Here $\mathbf{f}^{\text{int}}_v = (f^{\text{int}}_{1,v}, \ldots, f^{\text{int}}_{n-1,v})$, where $f^{\text{int}}_{k,v}$ denotes $\partial f^{\text{int}}_k/\partial v$, and $O(2)$ denotes terms of order 2 in ε or h as in (c). In the rest of this proof, we drop the dependence of the functions on $\mathbf{x}_{\text{land}}$ for simplicity.

Since $\mathbf{x}_{\text{land}}$ is in the sliding region, it follows that $\mathbf{x}'_b$ is $\varepsilon - h\varepsilon\frac{g^{\text{int}}_v}{g^{\text{int}}} + O(2)$ above the sliding region. Through a similar estimation as in part (c), to first order in ε and h, the time for the perturbed trajectory to arrive at the sliding region is

$$\tau(h, \varepsilon) = \frac{\varepsilon - h\varepsilon\frac{g^{\text{int}}_v}{g^{\text{int}}}}{-g^{\text{int}}} + O(2) = -\frac{\varepsilon}{g^{\text{int}}} + h\varepsilon\frac{g^{\text{int}}_v}{(g^{\text{int}})^2} + O(2).$$

At time τ, the location of the perturbed trajectory is

$$\begin{aligned}
\mathbf{x}'_{\text{land}} &= \Phi_{\text{II}}(\mathbf{x}'_b, \tau) \\
&= \mathbf{x}'_b + \tau\mathbf{F}^{\text{int}}(\mathbf{x}'_b) + O(2) \\
&= \mathbf{x}_{\text{land}} + \left(-h\varepsilon\frac{\mathbf{f}^{\text{int}}_v}{g^{\text{int}}}, \varepsilon - h\varepsilon\frac{g^{\text{int}}_v}{g^{\text{int}}}\right) + \tau\mathbf{F}^{\text{int}}(\mathbf{x}_{\text{land}}) + O(2) \\
&= \mathbf{x}_{\text{land}} + \left(-h\varepsilon\frac{\mathbf{f}^{\text{int}}_v}{g^{\text{int}}}, \varepsilon - h\varepsilon\frac{g^{\text{int}}_v}{g^{\text{int}}}\right) + \left(-\frac{\varepsilon}{g^{\text{int}}} + h\varepsilon\frac{g^{\text{int}}_v}{(g^{\text{int}})^2}\right)(\mathbf{f}^{\text{int}}, g^{\text{int}}) + O(2) \\
&= \mathbf{x}_{\text{land}} + (-h\varepsilon\frac{\mathbf{f}^{\text{int}}_v}{g^{\text{int}}}, 0) + (-\frac{\varepsilon}{g^{\text{int}}} + h\varepsilon\frac{g^{\text{int}}_v}{(g^{\text{int}})^2})(\mathbf{f}^{\text{int}}, 0) + O(2).
\end{aligned} \tag{SM3.20}$$

Simultaneously, the location of the unperturbed trajectory is

$$\begin{aligned} \mathbf{x}_\mathrm{c} &= \Phi_\mathrm{I}(\mathbf{x}_\mathrm{land}, \tau) \\ &= \mathbf{x}_\mathrm{land} + \tau \mathbf{F}^\mathrm{slide}(\mathbf{x}_\mathrm{land}) + O(2) \\ &= \mathbf{x}_\mathrm{land} + \left(-\tfrac{\varepsilon}{g^\mathrm{int}} + h\varepsilon \tfrac{g_v^\mathrm{int}}{(g^\mathrm{int})^2}\right)(\mathbf{f}^\mathrm{int}, 0) + O(2). \end{aligned} \tag{SM3.21}$$

Comparing the difference between (SM3.20) and (SM3.21), we see that

$$\left\|\mathbf{x}'_\mathrm{land} - \mathbf{x}_c\right\| = O(h\varepsilon). \tag{SM3.22}$$

Recall that the asymptotic phase is assumed to be C^1, with respect to displacements tangent to Σ. Since $\mathbf{x}'_\mathrm{land}$ and $\mathbf{x}_c$ are on Σ, it follows that

$$\phi(\mathbf{x}'_\mathrm{land}) - \phi(\mathbf{x}_c) = O(h\varepsilon).$$

Therefore, by (SM3.16),

$$\mathbf{z}_v(\mathbf{x}_a) = \lim_{\varepsilon\to 0} \frac{\phi(\mathbf{x}'_a) - \phi(\mathbf{x}_a)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\phi(\mathbf{x}'_\mathrm{land}) - \phi(\mathbf{x}_c)}{\varepsilon} = O(h)$$

Consequently,

$$\lim_{h\to 0} \mathbf{z}_v(\mathbf{x}_a) = 0$$

as required. This completes the proof of part (d).

(e) *The tangential components of* $\mathbf{z}$ *are continuous at both landing and liftoff points.* We denote the tangential components of the iPRC by $\mathbf{z}_\mathbf{w}$, where $\mathbf{w}$ represents vectors in the $n-1$ dimensional tangent space of the hard boundary. The $n-1$ dimensional iPRC vector $\mathbf{z}_\mathbf{w}$ obeys a restricted (*i.e.* reduced dimension) adjoint equation given in terms of $f_\mathbf{w}$, the $(n-1)\times(n-1)$ Jacobian derivative of f with respect to the $n-1$ tangential coordinates ($\mathbf{w}$), and $g_\mathbf{w}$, the $1\times(n-1)$ Jacobian derivative of g with respect to the tangential coordinates, and $\mathbf{z}_v$, the (scalar) component of $\mathbf{z}$ in the normal direction

$$\tfrac{d\mathbf{z}_\mathbf{w}}{dt} = -f_\mathbf{w}(\mathbf{w}, v)^\intercal \mathbf{z}_\mathbf{w} - g_\mathbf{w}(\mathbf{w}, v)^\intercal \mathbf{z}_v \tag{SM3.23}$$

along the limit cycle in the interior domain. On the other hand, along the sliding component of the limit cycle that is restricted to $\{\Sigma : v = 0\}$, $\mathbf{z}_\mathbf{w}$ satisfies

$$\tfrac{d\mathbf{z}_\mathbf{w}}{dt} = -f_\mathbf{w}(\mathbf{w}, 0)^\intercal \mathbf{z}_\mathbf{w}. \tag{SM3.24}$$

By part (c), $\mathbf{z}_v$ goes continuously to zero as the trajectory from the interior approaches the landing point. Therefore $\mathbf{z}_\mathbf{w}$ is continuous at the landing point.

Next we prove the continuity of $\mathbf{z}_\mathbf{w}$ at the liftoff point $\mathbf{x}_\mathrm{lift} = (\mathbf{w}_\mathrm{lift}, 0)$. Recall that in the coordinates employed, the unit vector tangent to Σ and normal to $\mathcal{L}$ at $\mathbf{x}_\mathrm{lift}$ is $\ell = (0, \ldots, 0, 1, 0)$ (cf. Fig. SM2). Fix an arbitrary tangential unit vector $\hat{\mathbf{w}}$ oriented away from the sliding region (such that $\ell^\intercal \hat{\mathbf{w}} > 0$). The left and right limits of $\mathbf{z}_\mathbf{w}$ at $\mathbf{x}_\mathrm{lift}$ are given by

$$\mathbf{z}^-_{\hat{\mathbf{w}}}(\mathbf{x}_\mathrm{lift}) = \lim_{\varepsilon\to 0^+} \frac{\phi(\mathbf{w}_\mathrm{lift} - \varepsilon\hat{\mathbf{w}}, 0) - \phi(\mathbf{w}_\mathrm{lift}, 0)}{-\varepsilon} \tag{SM3.25}$$

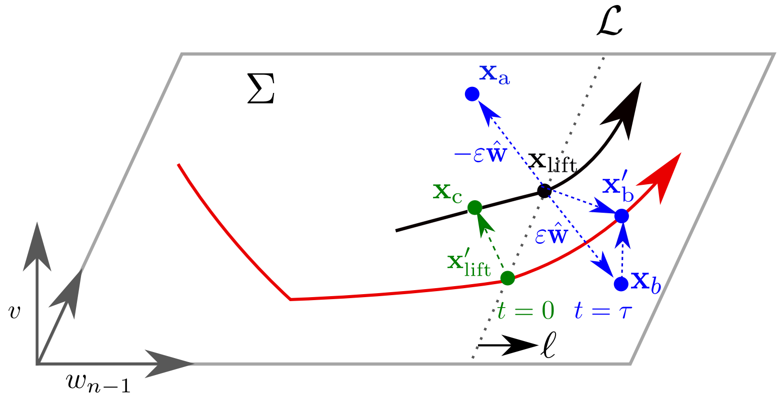

Figure SM2. *Unperturbed trajectory (black) leaves the hard boundary at the liftoff point $\mathbf{x}_{\text{lift}}$, in the $(\mathbf{w}, v)$ phase space. An instantaneous perturbation tangent to Σ is made to the liftoff point at $t = \tau$, pushing it to $\mathbf{x}_a$ on the sliding region or to $\mathbf{x}_b$ that is outside the sliding region. The points $\mathbf{x}_c$ and $\mathbf{x}'_{\text{lift}}$ denote the positions of the unperturbed trajectory and the perturbed trajectory at $t = 0$.*

and

$$\mathbf{z}^+_{\hat{\mathbf{w}}}(\mathbf{x}_{\text{lift}}) = \lim_{\varepsilon \to 0^+} \frac{\phi(\mathbf{w}_{\text{lift}} + \varepsilon\hat{\mathbf{w}}, 0) - \phi(\mathbf{w}_{\text{lift}}, 0)}{\varepsilon}. \tag{SM3.26}$$

By $\mathbf{x}_a = (\mathbf{w}_{\text{lift}} - \varepsilon\hat{\mathbf{w}}, 0)$ and $\mathbf{x}_b = (\mathbf{w}_{\text{lift}} + \varepsilon\hat{\mathbf{w}}, 0)$ we denote the two points that are located at a distance of ε away from $\mathbf{x}_{\text{lift}}$ along the $-\hat{\mathbf{w}}$ and $\hat{\mathbf{w}}$ directions, respectively (cf. Fig. SM2). We will show that

$$\mathbf{z}^-_{\hat{\mathbf{w}}}(\mathbf{x}_{\text{lift}}) = \mathbf{z}^+_{\hat{\mathbf{w}}}(\mathbf{x}_{\text{lift}}). \tag{SM3.27}$$

The equality of these limits will establish that $\mathbf{z}_{\mathbf{w}}$ is continuous at the liftoff point.

First, we consider $\mathbf{z}^+_{\hat{\mathbf{w}}}(\mathbf{x}_{\text{lift}})$. Given $\hat{\mathbf{w}}$, there exists a unique point $\mathbf{x}'_{\text{lift}}$ at the liftoff boundary $\mathcal{L} \cap \Sigma$, and a time $\tau > 0$, such that the trajectory beginning from $\mathbf{x}'_{\text{lift}}$ at time 0 passes directly over $\mathbf{x}'_b$ at time τ, in the sense that $\Phi_{\text{II}}(\mathbf{x}'_{\text{lift}}, \tau) = (\mathbf{w}_b, h)$, where Φ_{II} is the flow operator in the complement of the sliding region, $h > 0$ is the “height” of $\mathbf{x}'_b$ above $\mathbf{x}_b$, and $\mathbf{w}_b$ is the coordinate vector along the tangent space of the hard boundary. Let $\mathbf{x}_{\text{lift}} = (\mathbf{w}_{\text{lift}}, 0)$ and $\mathbf{x}'_{\text{lift}} = (\mathbf{w}'_{\text{lift}}, 0)$. By our construction, $\mathbf{w}_b = \mathbf{w}_{\text{lift}} + \varepsilon\hat{\mathbf{w}}$. Hence, the location of the perturbed trajectory at time τ is

$$\begin{aligned}(\mathbf{w}_b, h) = (\mathbf{w}_{\text{lift}} + \varepsilon\hat{\mathbf{w}}, h) &= \Phi_{\text{II}}(\mathbf{x}'_{\text{lift}}, \tau) \\ &= (\mathbf{w}'_{\text{lift}}, 0) + (f^{\text{int}}(\mathbf{x}'_{\text{lift}}), 0)\tau + O(\tau^2) \\ &= (\mathbf{w}'_{\text{lift}} + f^{\text{int}}(\mathbf{x}'_{\text{lift}})\tau + O(\tau^2), O(\tau^2)).\end{aligned}$$

Hence

$$\mathbf{w}_{\text{lift}} - \mathbf{w}'_{\text{lift}} = f^{\text{int}}(\mathbf{x}'_{\text{lift}})\tau - \varepsilon\hat{\mathbf{w}} + O(\tau^2), \tag{SM3.28}$$

$$h = O(\tau^2), \tag{SM3.29}$$

and

$$\mathbf{w}_b - \mathbf{w}'_{\text{lift}} = f^{\text{int}}(\mathbf{x}'_{\text{lift}})\tau + O(\tau^2) \tag{SM3.30}$$

On the other hand,

$$\varepsilon\hat{\mathbf{w}} + (\mathbf{w}'_{\text{lift}} - \mathbf{w}_b) = \mathbf{w}_{\text{lift}} - \mathbf{w}'_{\text{lift}}.$$

By (SM3.28) and (SM3.30), the above equation becomes

$$\varepsilon\hat{\mathbf{w}} - f^{\text{int}}(\mathbf{x}'_{\text{lift}})\tau = f^{\text{int}}(\mathbf{x}'_{\text{lift}})\tau - \varepsilon\hat{\mathbf{w}} + O(\tau^2).$$

That is,

$$\varepsilon\hat{\mathbf{w}} = f^{\text{int}}(\mathbf{x}'_{\text{lift}})\tau + O(\tau^2).$$

Taking the inner product of both sides with the unit vector ℓ (normal to $\mathcal{L}$), and noting that for sufficiently small ε, $\ell^\intercal f^{\text{int}}(x'_{\text{lift}}) > 0$ (our nondegeneracy condition), we have

$$\tau = \varepsilon\frac{\ell^\intercal\hat{\mathbf{w}}}{\ell^\intercal f^{\text{int}}(\mathbf{x}'_{\text{lift}})} + O(\tau^2),$$

and hence $\tau = O(\varepsilon)$. Therefore, (SM3.29) becomes

$$h = O(\varepsilon^2) \tag{SM3.31}$$

and hence the phase difference between $\mathbf{x}'_b$ and $\mathbf{x}_b$ is

$$\phi(\mathbf{x}_b) - \phi(\mathbf{x}'_b) = O(\varepsilon^2) \tag{SM3.32}$$

due to the assumption that ϕ is Lipschitz continuous.

Next we show (SM3.27) holds using (SM3.25) and (SM3.26). Let the unperturbed trajectory pass through $\mathbf{x}_{\text{lift}}$ at time τ, and let $\mathbf{x}_c$ be the location of the unperturbed trajectory at time $t = 0$ (see Fig. SM2). Let $\Delta\mathbf{x}_c = \mathbf{x}'_{\text{lift}} - \mathbf{x}_c$ and $\Delta\mathbf{x}_b = \mathbf{x}'_b - \mathbf{x}_{\text{lift}}$. Then by part (b),

$$\Delta\mathbf{x}_b - \Delta\mathbf{x}_c = O(|\Delta\mathbf{x}_b|^2);$$

since the saltation matrix is equal to the identity matrix at the liftoff boundary. Since $\mathbf{x}_{\text{lift}}, \mathbf{x}_b, \mathbf{x}'_b$ form a right triangle,

$$|\Delta\mathbf{x}_b|^2 = \varepsilon^2 + h^2 = \varepsilon^2 + O(\varepsilon^4),$$

which implies that

$$\Delta\mathbf{x}_b - \Delta\mathbf{x}_c = O(\varepsilon^2). \tag{SM3.33}$$

Direct computation shows

$$\begin{aligned}
\phi(\mathbf{x}_b) - \phi(\mathbf{x}_{\text{lift}}) &= (\phi(\mathbf{x}_b) - \phi(\mathbf{x}'_b)) + (\phi(\mathbf{x}'_b) - \phi(\mathbf{x}_{\text{lift}}))\\
&= (\phi(\mathbf{x}'_{\text{lift}}) - \phi(\mathbf{x}_c)) + O(\varepsilon^2)\\
&= D_{\mathbf{w}}\phi(\mathbf{x}_c)\cdot\Delta\mathbf{x}_c + O(\varepsilon^2)\\
&= D_{\mathbf{w}}\phi(\mathbf{x}_c)\cdot\Delta\mathbf{x}_b + O(\varepsilon^2)\\
&= D_{\mathbf{w}}\phi(\mathbf{x}_c)\cdot(\mathbf{x}'_b - \mathbf{x}_{\text{lift}}) + O(\varepsilon^2)\\
&= D_{\mathbf{w}}\phi(\mathbf{x}_c)\cdot(\mathbf{x}_b - \mathbf{x}_{\text{lift}}) + O(\varepsilon^2)\\
&= D_{\mathbf{w}}\phi(\mathbf{x}_c)\cdot\varepsilon\hat{\mathbf{w}} + O(\varepsilon^2).
\end{aligned} \tag{SM3.34}$$

To obtain the second equality, we translate the trajectories backward in time by τ beginning from $\mathbf{x}_b'$ and $\mathbf{x}_{\text{lift}}$, respectively; shifting both trajectories by an equal time interval does not change their phase relationship. The $O(\varepsilon^2)$ difference arises from (SM3.32). The third equality follows from the assumption that ϕ is differentiable with respect to displacements tangent to the sliding region. The fourth equality uses (SM3.33); the fifth and seventh follow from the definitions; the sixth uses (SM3.31).

Recall that we assume ϕ to have Lipschitz continuous derivatives in the tangential directions at the boundary surface (except possibly at the landing and liftoff points). Under this assumption, taking the limit $\varepsilon \to 0^+$ leads to $\mathbf{x}_c \to \mathbf{x}_{\text{lift}}^-$ and hence

$$\mathbf{z}_{\hat{\mathbf{w}}}^+(\mathbf{x}_{\text{lift}}) = D_{\mathbf{w}}\phi(\mathbf{x}_{\text{lift}}^-)\cdot\hat{\mathbf{w}}$$

by (SM3.26). On the other hand,

$$\begin{aligned}\phi(\mathbf{x}_a) - \phi(\mathbf{x}_{\text{lift}}) &= D_{\mathbf{w}}\phi(\mathbf{x}_a)\cdot(\mathbf{x}_a - \mathbf{x}_{\text{lift}}) + O(\varepsilon^2)\\ &= -D_{\mathbf{w}}\phi(\mathbf{x}_a)\cdot\varepsilon\hat{\mathbf{w}} + O(\varepsilon^2).\end{aligned} \tag{SM3.35}$$

Taking the limit $\varepsilon \to 0+$ results in $\mathbf{x}_a \to \mathbf{x}_{\text{lift}}^-$ and hence (SM3.25) together with (SM3.35), implies

$$\mathbf{z}_{\hat{\mathbf{w}}}^-(\mathbf{x}_{\text{lift}}) = D_{\mathbf{w}}\phi(\mathbf{x}_{\text{lift}}^-)\cdot\hat{\mathbf{w}}.$$

Hence, (SM3.27) holds. ■

SM4. Numerical Algorithms.

We will now describe how the results presented in section 2 and section 3 can be implemented as numerical algorithms.

Consider a multiple-zone Filippov system generalized from (3.4),

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}), \tag{SM4.1}$$

that produces a T_0-periodic limit cycle solution $\gamma(t) \subset \mathbf{R}^n$. Suppose $\gamma(t)$ includes k sliding components confined to boundary surfaces denoted as $\Sigma^i \subset \mathbf{R}^{n-1}$, $i \in \{1, ..., k\}$. $\gamma(t)$ exits the i-th boundary Σ^i at a unique liftoff point $\mathbf{x}_{\text{lift}}^i$ given that the nondegeneracy condition (3.6) at $\mathbf{x}_{\text{lift}}^i$ is satisfied. We denote the normal vector to Σ^i at liftoff, landing, or boundary crossing points by n^i. We denote the interior domain by $\mathcal{R}^{\text{interior}}$, which can now consist of multiple subdomains separated by transversal crossing boundaries, and denote the piecewise smooth vector field in $\mathcal{R}^{\text{interior}}$ by F^{interior}. By (3.4), the sliding vector field on the sliding region $\mathcal{R}^{\text{slide}_i} \subset \Sigma^i$ is therefore

$$F^{\text{slide}_i}(\mathbf{x}) = F^{\text{interior}}(\mathbf{x}) - (n^i \cdot F^{\text{interior}}(\mathbf{x}))n^i \tag{SM4.2}$$

Using this notation, the vector field (SM4.1) can be written as

$$F(\mathbf{x}) := \begin{cases} F^{\text{interior}}(\mathbf{x}), & \mathbf{x} \in \mathcal{R}^{\text{interior}} \\ F^{\text{slide}_i}(\mathbf{x}), & \mathbf{x} \in \mathcal{R}^{\text{slide}_i} \end{cases} \tag{SM4.3}$$

and we denote the vector field after a static perturbation by

$$F_\varepsilon(\mathbf{x}) := \begin{cases} F_\varepsilon^{\text{interior}}(\mathbf{x}), & \mathbf{x} \in \mathcal{R}^{\text{interior}} \\ F_\varepsilon^{\text{slide}_i}(\mathbf{x}), & \mathbf{x} \in \mathcal{R}^{\text{slide}_i} \end{cases} \tag{SM4.4}$$

where $i \in \{1, ..., k\}$. Here we assume that the regions are independent of static perturbation with size ε.

Notice that the computation of the iSRC requires estimating the rescaling factors, for which we need to compute the iPRC or the lTRC depending on whether a global uniform rescaling (2.20) or a piecewise uniform rescaling (2.26) is needed. We hence first present the numerical algorithms for obtaining the iPRC in section SM4.1 and the lTRC in section SM4.2; the algorithm for solving the homogeneous variational equation for the linear shape responses of $\gamma(t)$ to instantaneous perturbations (the variational dynamics $\mathbf{u}$) is presented in section SM4.3; lastly, in section SM4.4 we illustrate the algorithms for computing the linear shape responses of $\gamma(t)$ to sustained perturbations (the iSRC γ_1) with a uniform rescaling factor computed from the iPRC as well as with piecewise uniform rescaling factors computed from the lTRC.

For simplicity, we assume the initial time is $t_0 = 0$.

SM4.1. Algorithm for Calculating the iPRC $\mathbf{z}$ for LCSCs. It follows from Remark 3.15 that the iPRC $\mathbf{z}$ for the LCSCs need to be solved backward in time. While there is no discontinuity of $\mathbf{z}$ at a landing point, a time-reversed version of the jump matrix at the liftoff point on the hard boundary Σ^i, denoted as $\mathcal{J}^i_{\text{lift}}$, is given by

$$\mathcal{J}^i_{\text{lift}} = I - n^i {n^i}^\intercal, \tag{SM4.5}$$

where I is the identity matrix. $\mathcal{J}^i_{\text{lift}}$ updates $\mathbf{z}$ local to the liftoff point as

$$\mathbf{z}^{i-}_{\text{lift}} = \mathcal{J}^i_{\text{lift}} \mathbf{z}^{i+}_{\text{lift}} \tag{SM4.6}$$

where $\mathbf{z}^{i-}_{\text{lift}}$ and $\mathbf{z}^{i+}_{\text{lift}}$ are the iPRC just before and just after the trajectory crosses the liftoff point $x_{\text{lift},i}$ in forwards time.

We now describe an algorithm for numerically obtaining the complete iPRC $\mathbf{z}$ for $\gamma(t)$, a stable limit cycle with sliding components along hard boundaries and transversal crossing boundaries as described before.

Algorithm for $\mathbf{z}$.

1) Fix an initial condition $\mathbf{x}_0 = \gamma(0)$ on the limit cycle, and integrate (SM4.3) to compute $\gamma(t)$ over $[0, T_0]$.
2) Integrate the adjoint equation backward in time by defining $s = T_0 - t$ and numerically solve for the fundamental matrix $\Psi(s)$ over one period $0 \le s \le T_0$, where Ψ satisfies
   (a) $\Psi(0) = I$, the identity matrix.
   (b) For s such that $\gamma(T_0 - s)$ lies in the interior of the domain,
   $$\frac{d\Psi}{ds} = A^{\text{interior}}(T_0 - s)\Psi$$
   where $A^{\text{interior}}(t) = \left(DF^{\text{interior}}(\gamma(t))\right)^\intercal$ is the transpose of the Jacobian of the interior vector field F^{interior}.
   (c) For s such that $\gamma(T_0 - s)$ lies within a sliding component along boundary Σ^i,
   $$\frac{d\Psi}{ds} = A^i(T_0 - s)\Psi$$

where $A^i(t) = \big(DF^{\text{slide}_i}(\gamma(t))\big)^{\intercal}$ is the transpose of the Jacobian of the sliding vector field F^{slide_i}, given in (SM4.2).

(d) At any time t_p when γ transversely crosses a switching surface with a normal vector n_p,

$$\Psi^- = \mathcal{J}\Psi^+$$

where $\Psi^- = \lim_{s\to(T_0-t_p)^+}\Psi(s)$ and $\Psi^+ = \lim_{s\to(T_0-t_p)^-}\Psi(s)$ are the fundamental matrices just before and just after crossing the surface in forwards time. $\mathcal{J} = S^{\intercal}$ since $J^{\intercal}S = I$ as discussed in section 3.2, where the saltation matrix at any transversal crossing point is

$$S = I + \frac{(F_p^+ - F_p^-)n_p^{\intercal}}{n_p^{\intercal}F_p^-}$$

where F_p^-, F_p^+ are the vector fields just before and just after the crossing in forwards time (see (3.9)).

(e) At a liftoff point on the i-th hard boundary Σ^i (in backwards time, a transition from the interior to Σ^i), update Ψ as

$$\Psi^- = \mathcal{J}^i\Psi^+$$

where $\mathcal{J}^i = I - n^i n^{i\intercal}$ as defined in (SM4.5), and then switch the integration from the full Jacobian A^{interior} to the restricted Jacobian A^i.

(f) At a landing point on the i-th hard boundary Σ^i (in backwards time, a transition from Σ^i to the interior) switch integration from the restricted Jacobian A^i to the full Jacobian A^{interior}; no other change in Ψ is needed.

3) Diagonalize the fundamental matrix at one period $\Psi(T_0)$; it should have a single eigenvector v with unit eigenvalue. The initial value for $\mathbf{z}_{\text{BW}}$ (represented in *backwards time*) at the point $\gamma(T_0) = \gamma(0) = \mathbf{x}_0$ is given by

$$\mathbf{z}_{\text{BW}}(0) = \frac{v}{F(\mathbf{x}_0)\cdot v}$$

4) The iPRC in backward time over $s \in [0, T_0]$ is given by $\mathbf{z}_{\text{BW}}(s) = \Psi(s)\mathbf{z}_{\text{BW}}(0)$ and is T_0-periodic. Equivalently, one may repeat step (2) by replacing $\Psi(s)$ with $\mathbf{z}_{\text{BW}}(s)$ and replacing the initial condition $\Psi(0) = I$ with $\mathbf{z}_{\text{BW}}(0)$ to solve for the complete iPRC.
5) The iPRC in forward time is then given by $\mathbf{z}(t) = \mathbf{z}_{\text{BW}}(T_0 - t)$ where $t \in [0, T_0]$.
6) The linear shift in period in response to the static perturbation can be calculated by evaluating the integral (see (2.14))

$$T_1 = -\int_0^{T_0} \mathbf{z}^{\intercal}(t)\frac{\partial F_\varepsilon(\gamma(t))}{\partial\varepsilon}\Big|_{\varepsilon=0} dt$$

Remark SM4.1. *An alternative way (in MATLAB) to do backward integration is reversing the time span in the numerical solver; that is, integrate the adjoint equation over* $[T_0, 0]$ *to compute* $\mathbf{z}(t)$.

SM4.2. Algorithm for Calculating the lTRC for LCSCs. The lTRC satisfies the same adjoint equation, (2.9), as the iPRC, and hence exhibits the same jump matrix at each liftoff, landing and boundary crossing point. It follows that the algorithm for the iPRC from section SM4.1 can mostly carry over to computing the lTRC.

Suppose the domain of $\gamma(t)$ can be divided into m regions $\mathcal{R}^1, ..., \mathcal{R}^m$, each distinguished by its own timing sensitivity properties. We denote the lTRC in $\mathcal{R}^j$ by η^j.

Below we describe the algorithm to compute η^j in region $\mathcal{R}^j$ bounded by the two local timing surfaces Σ^{in} and Σ^{out}. Following the notations in section 2, t^{in} and t^{out} denote the time of entry into and exit out of $\mathcal{R}^j$, at locations $\mathbf{x}^{\text{in}}$ and $\mathbf{x}^{\text{out}}$, respectively. The algorithm for computing η^j is described as follows.

Algorithm for η^j.

1) Compute γ, the unperturbed limit cycle, and T_0, its period, by integrating (SM4.3).
2) Compute $t^{\text{in}}, t^{\text{out}}$ for region j. Evaluate $\mathbf{x}^{\text{in}} = \gamma(t^{\text{in}})$, $\mathbf{x}^{\text{out}} = \gamma(t^{\text{out}})$ and $T_0^j = t^{\text{out}} - t^{\text{in}}$.
3) Compute the boundary value for η^j at the exit point $\mathbf{x}^{\text{out}}$ (see (2.25))

$$\eta^j(\mathbf{x}^{\text{out}}) = \frac{-n^{\text{out}}}{n^{\text{out}\intercal} F(\mathbf{x}^{\text{out}})}$$

   where n^{out} is a normal vector to Σ^{out}.
4) Integrate the adjoint equation backward in time by defining $s = T_0 - t$ and numerically solve for $\eta^j_{\text{BW}}(s)$ (represented in backwards time) over $[T_0 - t^{\text{out}}, T_0 - t^{\text{in}}]$. $\eta^j_{\text{BW}}(s)$ satisfies the initial condition $\eta^j_{\text{BW}}(T_0 - t_{\text{out}}) = \eta^j(t_{\text{out}})$ computed from step (3) as well as conditions (b) through (f) from step (2) of **Algorithm for z** in section SM4.1.
5) The lTRC in forward time is then given by $\eta^j(t) = \eta^j_{\text{BW}}(T_0 - t)$ where $t \in [t_{\text{in}}, t_{\text{out}}]$.
6) Compute γ_ε, the limit cycle under some small static perturbation $\varepsilon \ll 1$, and find $\mathbf{x}^{\text{in}}_\varepsilon$, the coordinate of the intersection point where $\gamma_\varepsilon(t)$ crosses Σ^{in}. The linear shift in time in region j in response to the static perturbation can be calculated by evaluating the integral (see (2.23))

$$T_1^j = \eta^j(\mathbf{x}^{\text{in}}) \cdot \frac{\mathbf{x}^{\text{in}}_\varepsilon - \mathbf{x}^{\text{in}}}{\varepsilon} + \int_{t^{\text{in}}}^{t^{\text{out}}} \eta^j(\gamma(t)) \cdot \frac{\partial F_\varepsilon(\gamma(t))}{\partial \varepsilon}\Big|_{\varepsilon=0} dt.$$

Remark SM4.2. *All the local linear shifts in time sum up to the global linear shift in period, that is, $T_1 = \sum_{j=1}^{j=m} T_1^j$.*

SM4.3. Algorithm for Solving the Homogeneous Variational Equation for LCSCs. Here we describe the algorithm for solving the homogeneous variational equation for linear displacement $\mathbf{u}$, the shape response to an instantaneous perturbation. This makes use of Theorem 3.13, which describes different jumping behaviors of $\mathbf{u}$ at liftoff, landing, and boundary crossing points. Unlike the iPRC and lTRC which require integration backwards in time, the variational dynamics can be solved with forward integration. This makes the algorithm comparatively simpler by allowing $\gamma(t)$ and $\mathbf{u}(t)$ to be solved simultaneously.

Algorithm for $\mathbf{u}$:.

1) Fix an initial condition $\mathbf{x}_0 = \gamma(0)$ on the limit cycle and an initial condition $\mathbf{u}_0 = \mathbf{u}(0)$ for the displacement at $\gamma(0)$ of the limit cycle.

2) Integrate the original differential equation (SM4.3) and the homogeneous variational equation (2.7) simultaneously forward in time and numerically solve for $\mathbf{u}(t)$ over one period $0 \le t \le T_0$, where $\mathbf{u}$ satisfies
   (a) $\mathbf{u}(0) = \mathbf{u}_0$.
   (b) For t such that $\gamma(t)$ lies in the interior of the domain,
$$\frac{d\mathbf{u}}{dt} = DF^{\text{interior}}(\gamma(t))\mathbf{u}$$
   (c) For t such that $\gamma(t)$ lies within a sliding component along boundary Σ^i,
$$\frac{d\mathbf{u}}{dt} = DF^{\text{slide}_i}(\gamma(t))\mathbf{u}$$
   where DF^{slide_i} is the Jacobian of the sliding vector field F^{slide_i} given in (SM4.2).
   (d) At any time t_p when γ transversely crosses a switching surface with a normal vector n_p separating vector field F_p^- on the incoming side from vector field F_p^+ on the outgoing side,
$$\mathbf{u}^+ = S\mathbf{u}^-$$
   where $\mathbf{u}^- = \lim_{t \to t_p^-} \mathbf{u}(t)$ and $\mathbf{u}^+ = \lim_{t \to t_p^+} \mathbf{u}(t)$ are the displacements just before and just after crossing the surface. By the definition for the saltation matrix at transversal crossing point (3.9), we have
$$S = I + \frac{(F_p^+ - F_p^-)n_p^\intercal}{n_p^\intercal F_p^-}.$$
   (e) At a landing point on the i-th hard boundary Σ^i, update $\mathbf{u}$ as
$$\mathbf{u}^+ = S^i\mathbf{u}^-$$
   where $S^i = I - n^i n^{i\intercal}$ (recall n^i is the normal vector to Σ^i) and switch integration from the full Jacobian DF^{interior} to the restricted Jacobian DF^{slide_i}.
   (f) At a liftoff point on the i-th hard boundary Σ^i, switch integration from the restricted Jacobian DF^{slide_i} to the full Jacobian DF^{interior}; no other change in $\mathbf{u}$ is needed.

Remark SM4.3. *The fundamental solution matrix satisfies*
$$\frac{d\Phi(t,0)}{dt} = DF\Phi(t,0), \text{ with } \quad \Phi(0,0) = I$$
and takes the initial perturbation $\mathbf{u}(0)$ to the perturbation $\mathbf{u}(t)$ at time t, that is,
$$\mathbf{u}(t) = \Phi(t,0)\mathbf{u}(0).$$
*Computing Φ therefore requires applying **Algorithm for $\mathbf{u}$** n times, once for each dimension of the state space. Specifically, let $\Phi(t,0) = [\phi_1(t,0) \ldots, \phi_n(t,0)]$. The i-th column $\phi_i(t,0)$ is the solution of the variational equation (2.7) with the initial condition $\phi_i(0,0) = e_i$, a unit column vector with zeros everywhere except at the i-th row where the entry equals 1.*

Remark SM4.4. *Once Φ is obtained, we can obtain the monodromy matrix, $M = \Phi(T_0, 0)$. It follows from the periodicity of $\gamma(t)$ that M has $+1$ as an eigenvalue with eigenvector v tangent to the limit cycle at $\mathbf{x}_0$; this condition provides a partial consistency check for the algorithm.*

SM4.4. Algorithms for computing iSRC, the response to sustained perturbation. Now we discuss the calculation of iSRC γ_1, the linear shape response to a sustained perturbation. While γ_1 shares the same saltation matrix as $\mathbf{u}$ at each liftoff, landing and boundary crossing point, γ_1 satisfies the nonhomogeneous version of the variational equation, (2.20) or (2.26), where one of the nonhomogeneous terms depends on the time scaling factor, ν_1 or ν_1^j. Moreover, the initial condition for γ_1 depends on the given perturbation and hence needs to be computed in the algorithm whereas the initial value for $\mathbf{u}$ is arbitrarily preassigned.

In the following, we first describe the algorithm for computing γ_1 using the global uniform rescaling and then consider using piecewise uniform rescaling.

Algorithm for γ_1 with uniform rescaling.

1) Fix an initial condition $\mathbf{x}_0 = \gamma(0)$ on the limit cycle.
2) Compute the linear shift in period T_1 using **Algorithm for z**, then evaluate $\nu_1 = T_1/T_0$.
3) Choose an arbitrary Poincaré section Σ (this can be one of the switching boundaries for appropriate $\mathbf{x}_0$) that is transverse to γ at $\mathbf{x}_0$. Compute γ_ε, the limit cycle under some fixed small static perturbation, and find $\mathbf{x}_{0\varepsilon}$, the coordinate of the intersection point where $\gamma_\varepsilon(t)$ crosses Σ. The initial value for γ_1 at the initial point $\mathbf{x}_0$ is then given by
$$\gamma_1(0) = \frac{\mathbf{x}_{0\varepsilon} - \mathbf{x}_0}{\varepsilon}$$
4) Integrate the original differential equation (SM4.3) with the initial condition $\mathbf{x}_0$ and the nonhomogeneous variational equation (2.20) simultaneously forward in time and numerically solve for γ_1 over one period $0 \le t \le T_0$, where γ_1 satisfies
   (i) $\gamma_1(0) = (\mathbf{x}_{0\varepsilon} - \mathbf{x}_0)/\varepsilon$.
   (ii) For t such that $\gamma(t)$ lies in the interior of the domain,
$$\frac{d\gamma_1}{dt} = DF^{\text{interior}}(\gamma(t))\gamma_1 + \nu_1 F^{\text{interior}}(\gamma(t)) + \frac{\partial F_\varepsilon^{\text{interior}}(\gamma(t))}{\partial \varepsilon}\Big|_{\varepsilon=0}$$
   (iii) For t such that $\gamma(t)$ lies within a sliding component along boundary Σ^i,
$$\frac{d\gamma_1}{dt} = DF^{\text{slide}_i}(\gamma(t))\gamma_1 + \nu_1 F^{\text{slide}_i}(\gamma(t)) + \frac{\partial F_\varepsilon^{\text{slide}_i}(\gamma(t))}{\partial \varepsilon}\Big|_{\varepsilon=0}$$
   where DF^{slide_i} is the Jacobian of the sliding vector field F^{slide_i} given in (SM4.2).
   (iv) For transversal crossings, landing points, and liftoff points, apply (d), (e) and (f), respectively, from step 2) in **Algorithm for u** in section SM4.3, by replacing $\mathbf{u}$ with γ_1.

Next we consider the case when $\gamma(t)$ exhibits m different uniform timing sensitivities at regions $\mathcal{R}^1, \ldots, \mathcal{R}^m$, each bounded by two local timing surfaces, as discussed in section SM4.2. Piecewise uniform rescaling is therefore needed to compute the shape response curve. The procedure for obtaining γ_1 in this case is nearly the same as described in **Algorithm for** γ_1 **with uniform rescaling**, except we now need to compute various rescaling factors using the lTRC. This hence leads to different variational equations that need to be solved. On the other hand, the local timing surfaces naturally serve as the Poincaré sections that are required to compute the initial values for γ_1 in the uniform rescaling case.

Algorithm for γ_1 with piecewise uniform rescaling.

1) Take the initial condition for $\gamma(t)$ to be $\gamma(0) = \mathbf{x}_0 \in \Sigma$, where Σ is one of the local timing surfaces. Compute $\gamma(t)$, the unperturbed trajectory, and $\gamma_\varepsilon(t)$, the trajectory under some static perturbation $0 < \varepsilon \ll 1$, by integrating (SM4.3).
2) For $j \in \{1, ..., m\}$, compute T_0^j, the time that $\gamma(t)$ spends in region j and T_1^j, the linear shift in time in region j using **Algorithm for** η^j, and then evaluate $\nu_1^j = T_1^j / T_0^j$.
3) Compute $\mathbf{x}_{0\varepsilon}$, the coordinate of the intersection point where $\gamma_\varepsilon(t)$ crosses Σ. The initial value for γ_1 at the initial point $\mathbf{x}_0$ is given by

$$\gamma_1(0) = \frac{\mathbf{x}_{0\varepsilon} - \mathbf{x}_0}{\varepsilon}.$$

4) Integrate the original differential equation (SM4.3) with the initial condition $\mathbf{x}_0$ and the piecewise nonhomogeneous variational equation (2.26) simultaneously forward in time and numerically solve for γ_1 over one period $0 \le t \le T_0$, where γ_1 satisfies
   (i) $\gamma_1(0) = (\mathbf{x}_{0\varepsilon} - \mathbf{x}_0)/\varepsilon$.
   (ii) For t such that $\gamma(t)$ lies in the intersection of the interior of the domain and region $\mathcal{R}^j$,

$$\frac{d\gamma_1}{dt} = DF^{\text{interior}_j}(\gamma(t))\gamma_1 + \nu_1^j F^{\text{interior}_j}(\gamma(t)) + \frac{\partial F_\varepsilon^{\text{interior}_j}(\gamma(t))}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

   where DF^{interior_j} is the Jacobian of the interior vector field F^{interior_j} in $\mathcal{R}^j$.
   (iii) For t such that $\gamma(t)$ lies within the intersection of a hard boundary Σ^i and region $\mathcal{R}^j$,

$$\frac{d\gamma_1}{dt} = DF^{\text{slide}_i}(\gamma(t))\gamma_1 + \nu_1^j F^{\text{slide}_i}(\gamma(t)) + \frac{\partial F_\varepsilon^{\text{slide}_i}(\gamma(t))}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

   where DF^{slide_i} is the Jacobian of the sliding vector field $F^{\text{slide}_i}(\mathbf{x}) = F^{\text{interior}_j}(\mathbf{x}) - (n^i \cdot F^{\text{interior}_j}(\mathbf{x}))n^i$ given in (SM4.2).
   (iv) For transversal crossings, landing points, and liftoff points, apply (d), (e) and (f), respectively, from step 2) in **Algorithm for u** in section SM4.3, replacing $\mathbf{u}$ with γ_1.

SM5. Table of Common Symbols.

Symbol	Meaning
$\mathbf{x}$	state variables
t	time
$\theta(t)$	phase of a limit cycle
$\phi(\mathbf{x})$	asymptotic phase of a stable limit cycle
$F(\mathbf{x})$	unperturbed velocity vector field
$\gamma(t)$	unperturbed limit cycle solution
T	period of the unperturbed limit cycle
εP	small instantaneous perturbation vector
$\tilde{\gamma}(t)$	trajectory near limit cycle after instantaneous perturbation
$\mathbf{u}(t) \simeq \tilde{\gamma}(t) - \gamma(t)$	displacement from limit cycle after instantaneous perturbation
ε	sustained (parametric) perturbation
$F_\varepsilon(\mathbf{x})$	perturbed velocity vector field
$\gamma_\varepsilon(t)$	perturbed limit cycles solution
T_ε	period of the perturbed limit cycle
$F_0 = F$	zeroth-order term of Taylor expansion of F_ε around $\varepsilon = 0$
$\gamma_0 = \gamma$	zeroth-order term of Taylor expansion of γ_ε around $\varepsilon = 0$
$T_0 = T$	zeroth-order term of Taylor expansion of T_ε around $\varepsilon = 0$
$F_1 = \partial F_\varepsilon / \partial \varepsilon \big\|_{\varepsilon=0}$	first-order term of Taylor expansion of F_ε around $\varepsilon = 0$
$\gamma_1 = \partial \gamma_\varepsilon / \partial \varepsilon \big\|_{\varepsilon=0}$	first-order term of Taylor expansion of γ_ε around $\varepsilon = 0$, also called the infinitesimal shape response curve (iSRC)
$T_1 = \partial T_\varepsilon / \partial \varepsilon \big\|_{\varepsilon=0}$	first-order term of Taylor expansion of T_ε around $\varepsilon = 0$
DF; $D_{\mathbf{w}}\phi$	Jacobian matrix; directional derivative of ϕ in $\mathbf{w}$ direction
I	identity matrix
S	saltation matrix (for variation equation)
J	jump matrix (for adjoint equation)
$\mathcal{J}$	time-reversed jump matrix (for adjoint equation)
Σ^i	boundary i
$\mathcal{R}^j$	region j
$F^j(\mathbf{x})$	velocity vector field in region j
$\nu_\varepsilon = T_0/T_\varepsilon$	relative frequency of perturbed limit cycle
$\nu_1 = T_1/T_0$	first-order term of Taylor expansion of ν_ε around $\varepsilon = 0$, also called the relative change in frequency
$\mathcal{T}^j$	time remaining in region j along a trajectory
$\mathbf{u}(t)$	variational dynamics governed by (2.7)
$\mathbf{z}(t) = \nabla_{\mathbf{x}}\phi(\gamma(t))$	infinitesimal phase response curve (iPRC) governed by (2.9)
$\gamma_1(t)$	infinitesimal shape response curve (iSRC) governed by (2.20)
$\eta^j(t) = \nabla_{\mathbf{x}}\mathcal{T}^j(\gamma(t))$	local timing response curve (lTRC) governed by (2.24)

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