10.5 Liapunov Exponent   Strogatz Book

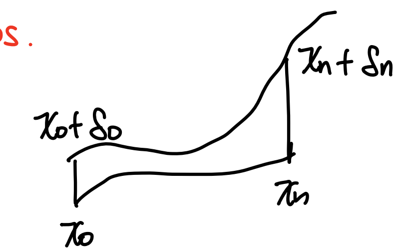
To study chaos, we use the Liapunov exponent to quantify sensitive dependence on initial conditions.

- Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$ where the initial separation δ_0 is extremely small $|\delta| \ll 1$.
- Let δ_n be the separation after n iterates.

If $|\delta_n| \approx |\delta_0| e^{n\lambda}$, then λ is called the Liapunov exponent.
(Exponential separation rate)

"To be called "chaotic", a system should also show sensitive dependence on initial conditions, in the sense that neighboring orbits separate exponentially fast, on average."

★ A positive Liapunov exponent is a signature of chaos.



How to find or compute λ? $x_{n+1} = f(x_n)$

$$e^{n\lambda} \approx \frac{|\delta_n|}{|\delta_0|} \Rightarrow \lambda \approx \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right|$$

Recall $x_n = f^n(x_0)$

$$x_n + \delta_n = f^n(x_0 + \delta_0)$$

$$\Rightarrow \delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$$

$$\therefore \lambda \approx \frac{1}{n} \ln \left| \frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0} \right|$$

$$g(x_0 + \delta_0) = g(x_0) + g'(x_0)\delta_0 + O(\delta_0^2) \quad \text{— Taylor Expansion}$$

$$\Rightarrow f^n(x_0+\delta_0) - f^n(x_0) = (f^n(x_0))'\delta_0 + O(\delta_0^2)$$

$$\therefore \lambda \approx \frac{1}{n} \ln |(f^n)'(x_0)| \quad \text{as } \delta_0 \to 0$$

$$(f^n)'(x_0) = (f \circ f \circ \cdots \circ f)'(x_0) = f'(f^{n-1}(x_0)) \cdot f'(f^{n-2}(x_0)) \cdots f'(x_0)$$

$$(f(f(x_0)))' = f'(f(x_0)) f'(x_0) = f'(x_1) f'(x_0)) \quad = f'(x_{n-1}) f'(x_{n-2}) \cdots f'(x_0)$$

$\therefore (f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$

Hence, $\lambda \approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \left(\ln|f'(x_{n-1})| + \ln|f'(x_{n-2})| + \cdots + \ln|f'(x_0)| \right)$

$\uparrow$ ($\ln|xy| = \ln|x||y| = \ln|x| + \ln|y|$)

$= \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$

Let $n \to \infty$, then $\lambda = \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \right\}$ is defined as the Liapunov exponent for the orbit starting at x_0.

* λ depends on the initial condition x_0, but it's the same for all x_0 in the basin of attraction of a given attractor.

For stable fixed points and cycles, $\lambda < 0$

For chaotic attractors, $\lambda > 0$.

what happens at fixed-pt bif ? (including SN, pitchfork, transcritical, PD)

consider Fixed point x^*, $f(x^*) = x^* \Rightarrow x_i = x^* \; \forall i$ if $x_0 = x^*$.

$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| = \ln|f'(x^*)|$

If bifurcation occurs at x^*, then $|f'(x^*)| = 1 \Rightarrow \lambda = \ln 1 = 0$

If x^* is stable, $|f'(x^*)| < 1 \Rightarrow \lambda < 0$.

Similarly, if f has a stable p-cycle containing x_0. The Liapunov exponent $\lambda < 0$.

<u>Proof</u>: x_0 is an element of a p-cycle

so x_0 is a fixed point of f^p & $|(f^p)'(x_0)| < 1$ ↙ multiplier of the cycle

$\therefore \ln|(f^p)'(x_0)| < 0$

For a p-cycle, $\lambda = \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \right\}$

$= \frac{1}{p} \sum_{i=0}^{p-1} \ln|f'(x_i)| = \frac{1}{p} \ln|(f^p)'(x_0)| < 0$ #

↑ as the same p terms keep repeating

sum of ln = ln of $\prod$ = ln of $(f^p)'$

Numerically, easy to find λ. Use for-loop to iterate the orbit from x_0 for n large enough, within for-loop, add up $\ln|f'(x_0)| + \ln|f'(x_1)| + \cdots + \ln|f'(x_{n-1})| = \text{sum}$

then $\lambda \simeq \text{sum}/n$ (better to get rid of transients)

Principles of Mathematical Modeling and Applications

November 2025

Dr. Yangyang Wang (yangyangwang@brandeis.edu)

Math Department, Brandeis University

MATH 123A – Chaos

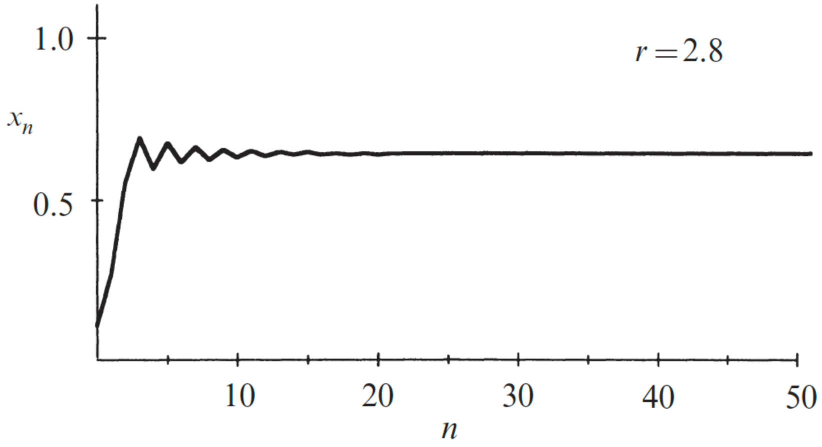
Butterfly Effect – Chaos Theory

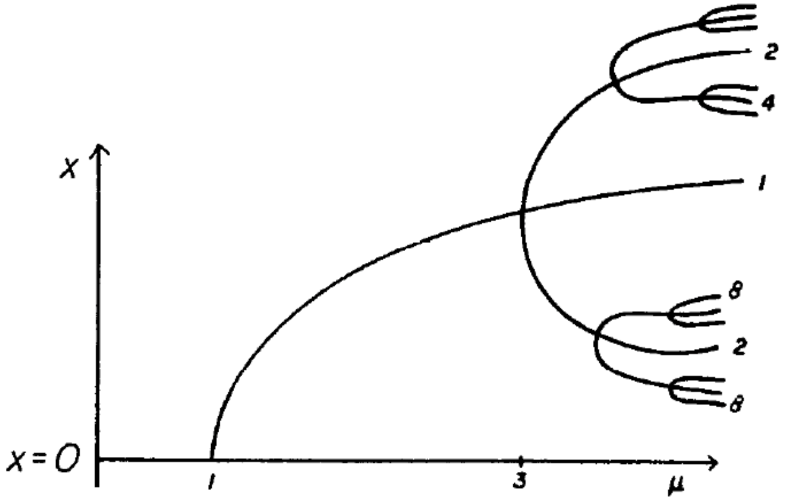
https://www.youtube.com/watch?v=r_ahZOgPTsk

Logistic Map $x_{n+1} = rx_n(1 - x_n)$

$0 < r < 1$: Population always goes extinct

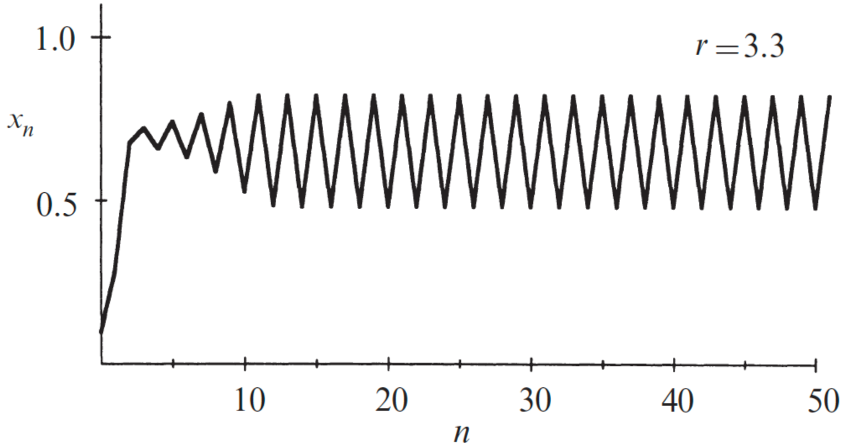
$1 < r < 3$:



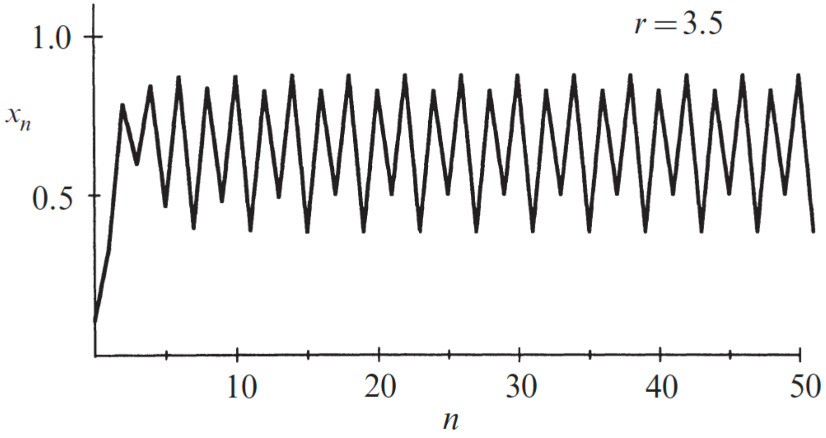


$r > 3$

Period 2



Period 4

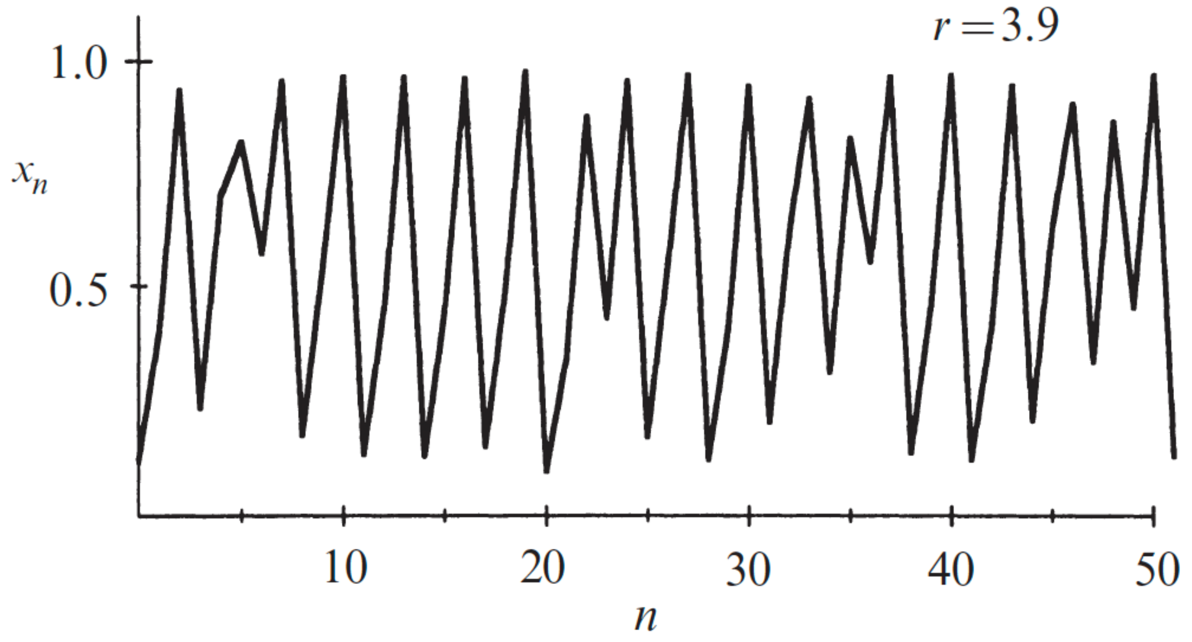


A cascade of periodic doubling bifurcations

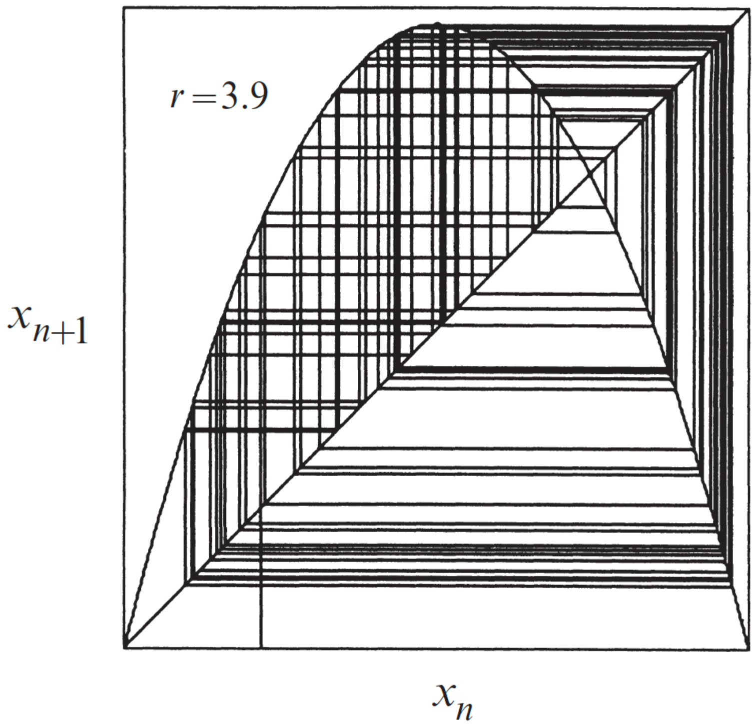
$r_1 = 3$	(period 2 is born)
$r_2 = 3.449\ldots$	4
$r_3 = 3.54409\ldots$	8
$r_4 = 3.5644\ldots$	16
$r_5 = 3.568759\ldots$	32
$\vdots$	$\vdots$
$r_\infty = 3.569946\ldots$	∞

What happens for $r > r_\infty$

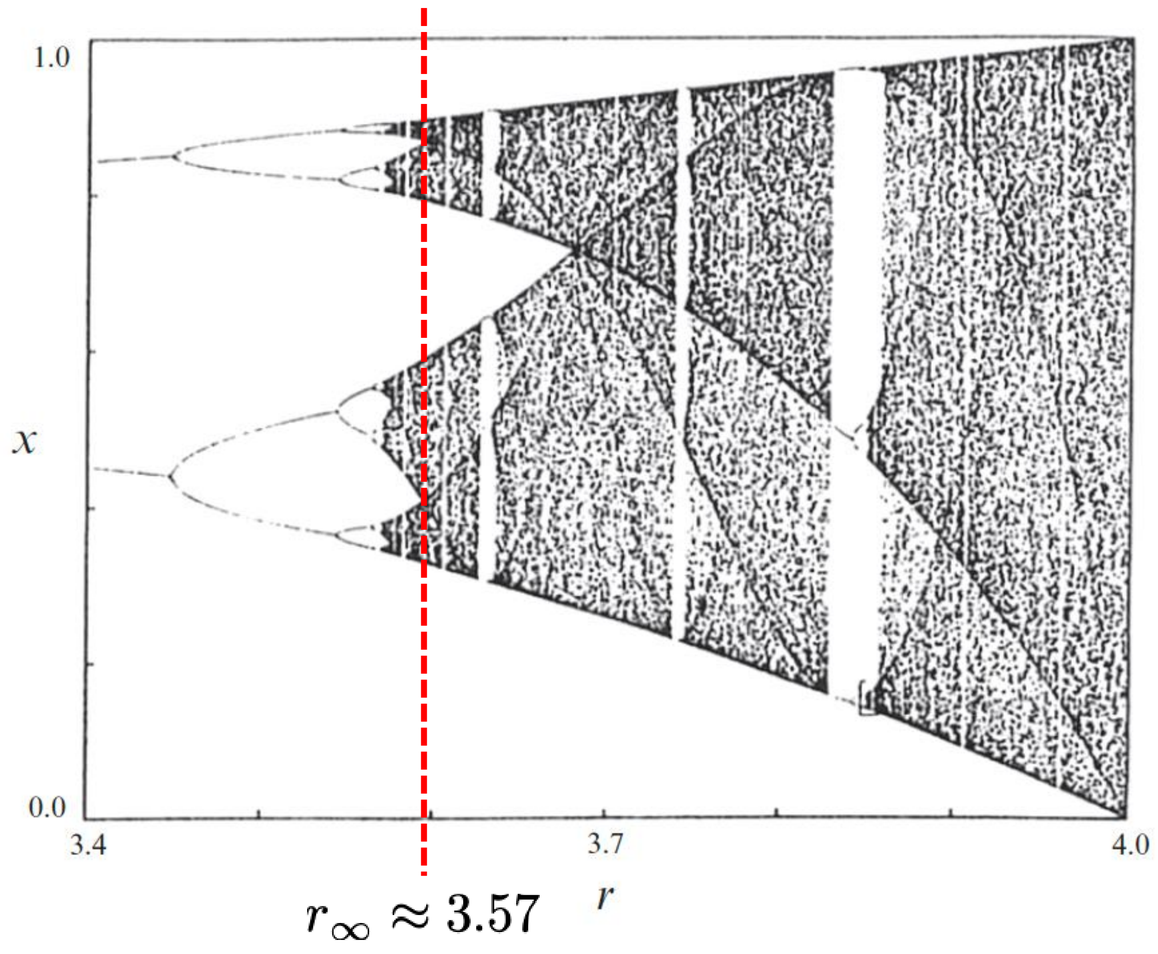
Time series



Poincare map/ Cobweb Diagram



Does the system become more and more chaotic as r increases?



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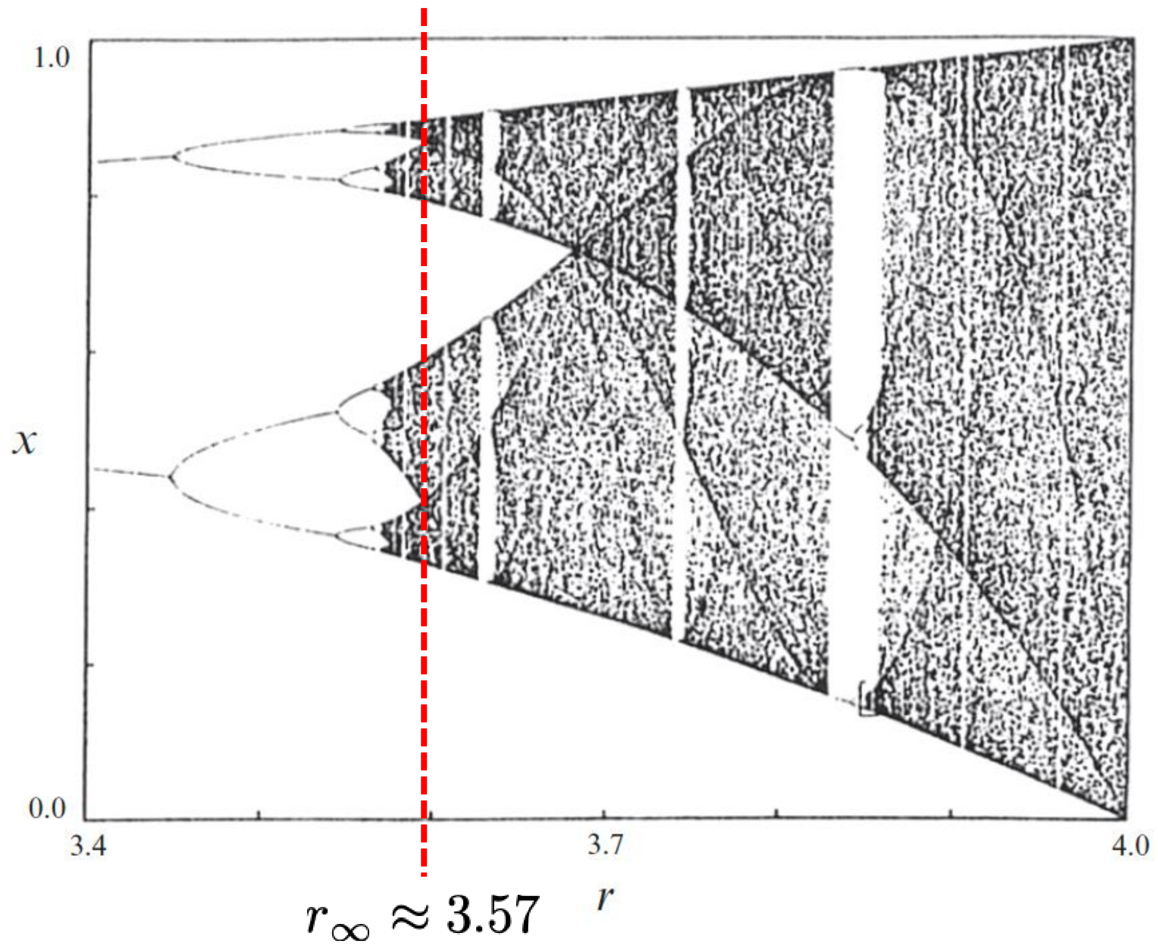

Chaos

- How do we know these aperiodic behaviors are really Chaos?

- To be called “Chaotic”, a system should also show sensitive dependence on initial conditions, in the sense that neighboring orbits separate exponentially fast

Liapunov Exponent