

Graph Theory & Networks

Math 238A: Topics in Applied Mathematics, Fall 2025

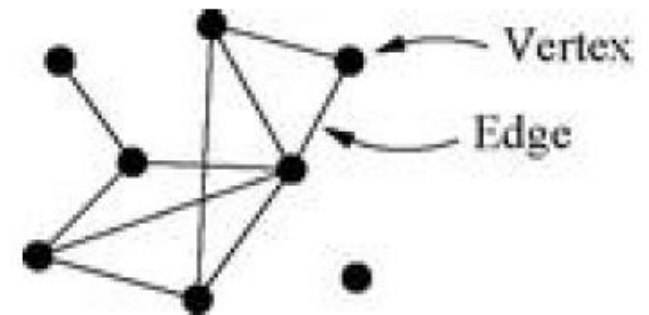
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Networks

A network, in its simplest form, is a collection of points (vertices/nodes) joined together in pairs by lines (edges).

Many objects of interest in the physical, biological, and social sciences can be thought of as networks



Why are we interested in networks?

- Wide applications:
 - Internet: a collection of computers linked by data connections
 - Human Societies: collections of people linked by acquaintance or social interaction

Network	Vertex	Edge
Internet	Computer or router	Cable or wireless data connection
World Wide Web	Web page	Hyperlink
Citation network	Article, patent, or legal case	Citation
Power grid	Generating station or substation	Transmission line
Friendship network	Person	Friendship
Metabolic network	Metabolite	Metabolic reaction
Neural network	Neuron	Synapse
Food web	Species	Predation

- Many aspects are worthy of study
 - Nature of the individual components – how a computer works or how a human being feels or acts
 - Nature of the connections or interactions – the communication protocols used on the Internet or the dynamics of human friendships
 - Pattern of connections between components – which can be represented as a **network**

Why are we interested in networks?

The structure of a network can have a big impact on the behavior of the corresponding system

- The connections in a social network affect how people learn, form opinions, and gather news, as well as affecting other less obvious phenomena, such as the spread of disease.
- Unless we know something about the structure of these networks, we cannot hope to understand fully how the corresponding systems work.
- A network is a simplified representation that reduces a system to an abstract structure capturing only the basics of connection patterns and little else.
- More information can be added (e.g., label coupling strength, directions), but even so a lot of information is lost when reducing a full system to a network representation. This has its disadvantages but it has advantages as well.

Why are we interested in networks?

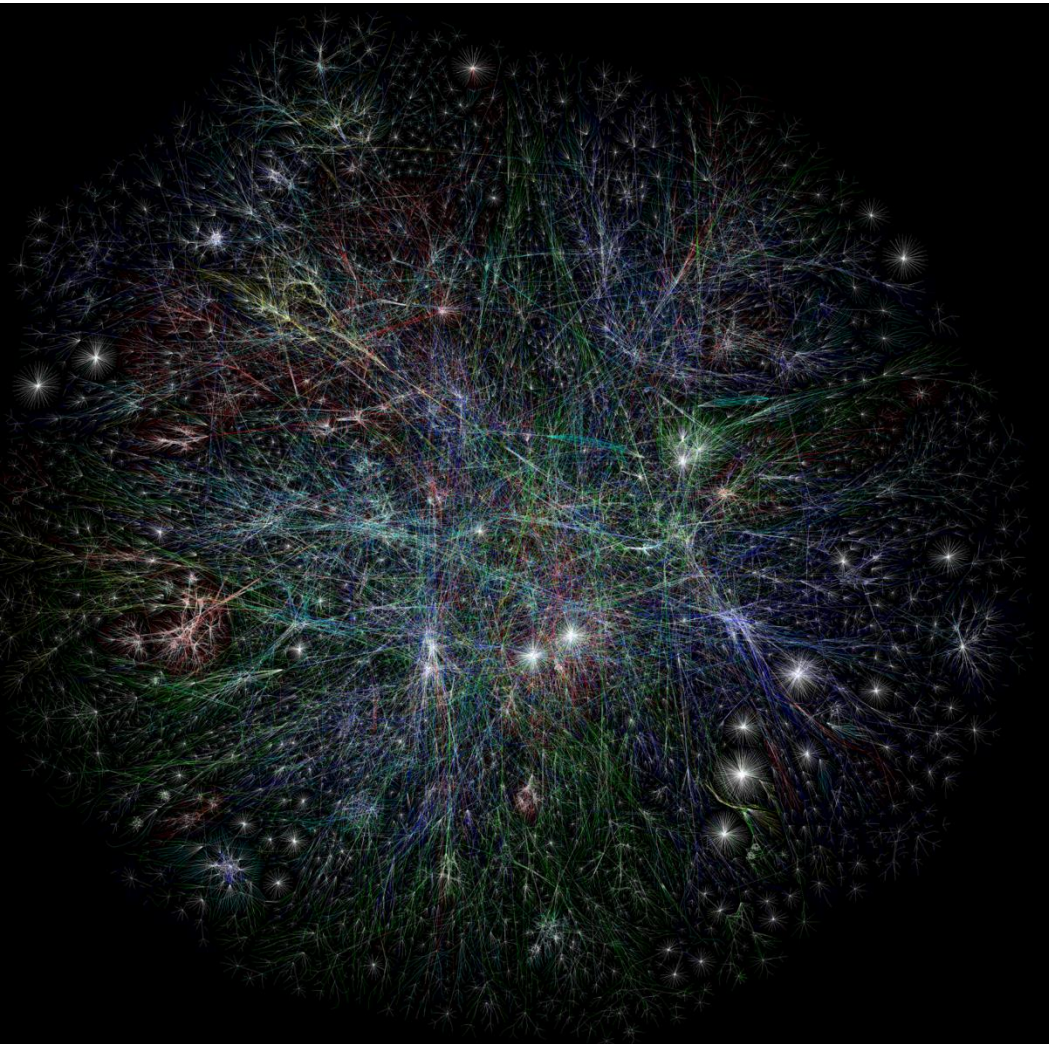
- Networks offer a **universal framework** for representing interactions in complex systems
- A wide range of tools—**mathematical, computational, and statistical**—already exist for analyzing, modeling and understanding networks. For example,
 - Identifying key nodes or connections
 - Measuring path lengths and connectivity
 - Modeling dynamic processes (e.g., traffic, disease spread)
- These tools work with networks in their abstract form and can in theory be applied to almost any system represented as a network:
 - Social networks, neural networks, metabolic pathways, power grids, the Internet, etc.

The power of the network Approach

- Field-independent: same tools apply across disciplines with minimal modification
- Supports both structural analysis and dynamical modeling
- Model-independent: can generate useful insights and predictions without requiring detailed equations or system-specific modeling
- Especially effective when paired with a clear and well-posed research question about a networked system

Some examples of networks

Some examples of networks



One of the best known and most widely studied examples of a network is the **Internet**.

In The Internet 2003 map:

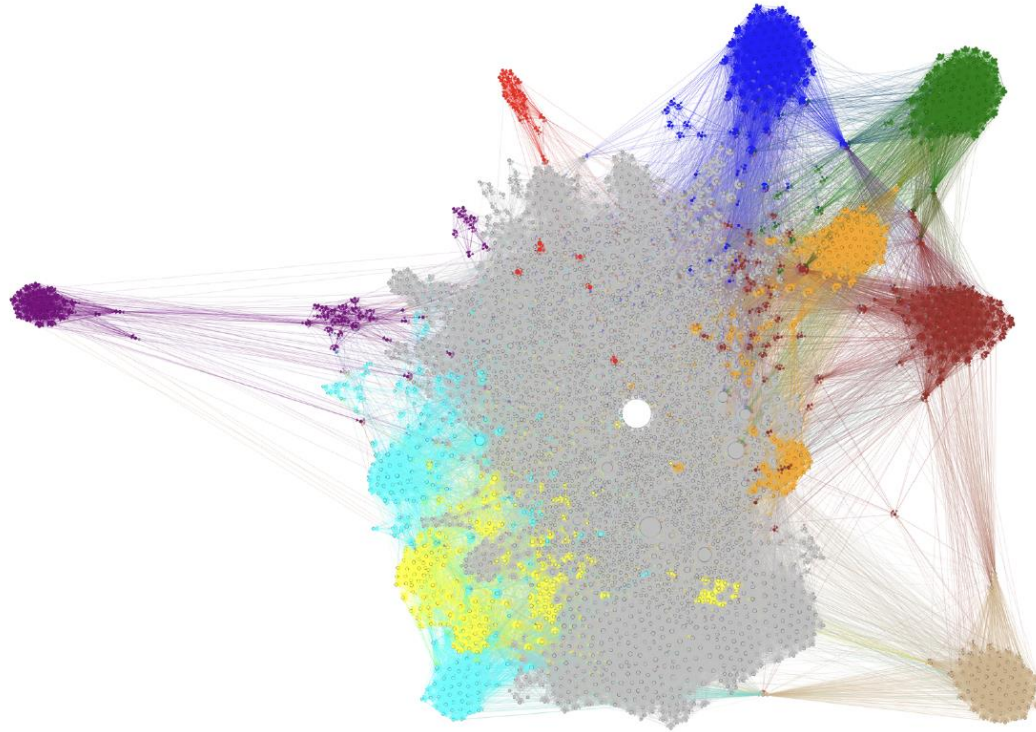
- Nodes/Vertices: “class C subnets” – groups of computers with similar IP addresses
- Edges/Connections: the routes taken by Internet data packets as they hop between subnets

Figure taken from the Opte Project (<https://www.opte.org/>).

World Wide Web

- A network of information stored on web pages.
 - Vertices: web pages
 - Edges: hyperlinks – the highlighted snippets of text or push-buttons on web pages that we click on to navigate from one page to another
- Enormously useful and profitable, with billions of pages and links
- The structure of the network of links is of substantial interest
- *Google* search engine sorts millions of web pages (nodes) matching a certain query based on
 - Similarity between page content and query (e.g., *Cosine Similarity*)
 - *PageRank* - a network centrality measure that captures importance of each node

World Wide Web: directed networks of information



A portion of the Wikipedia information network. Nodes are articles about math. We only consider links among Wikipedia articles, and disregard links to external pages. Node size is proportional to article importance, and colors highlight communities discussed in the text.

A small subset of articles (pages) about math within English Wikipedia

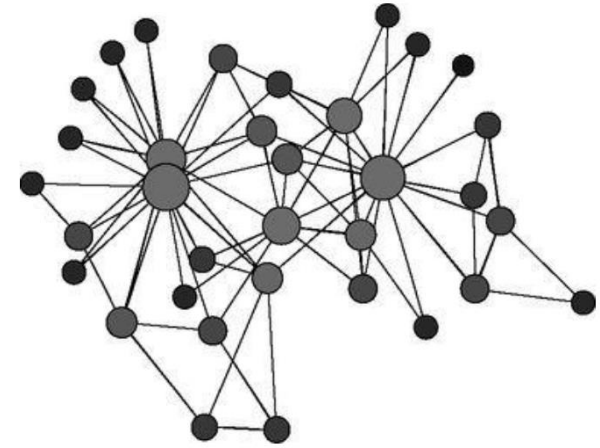
Node size represents PageRank

The large white node in the middle is the general article about Mathematics.

Existence of clusters of nodes on specific topics or branches of math: e.g., cyan – statistics; yellow – game theory ; purple – math software

Social Networks

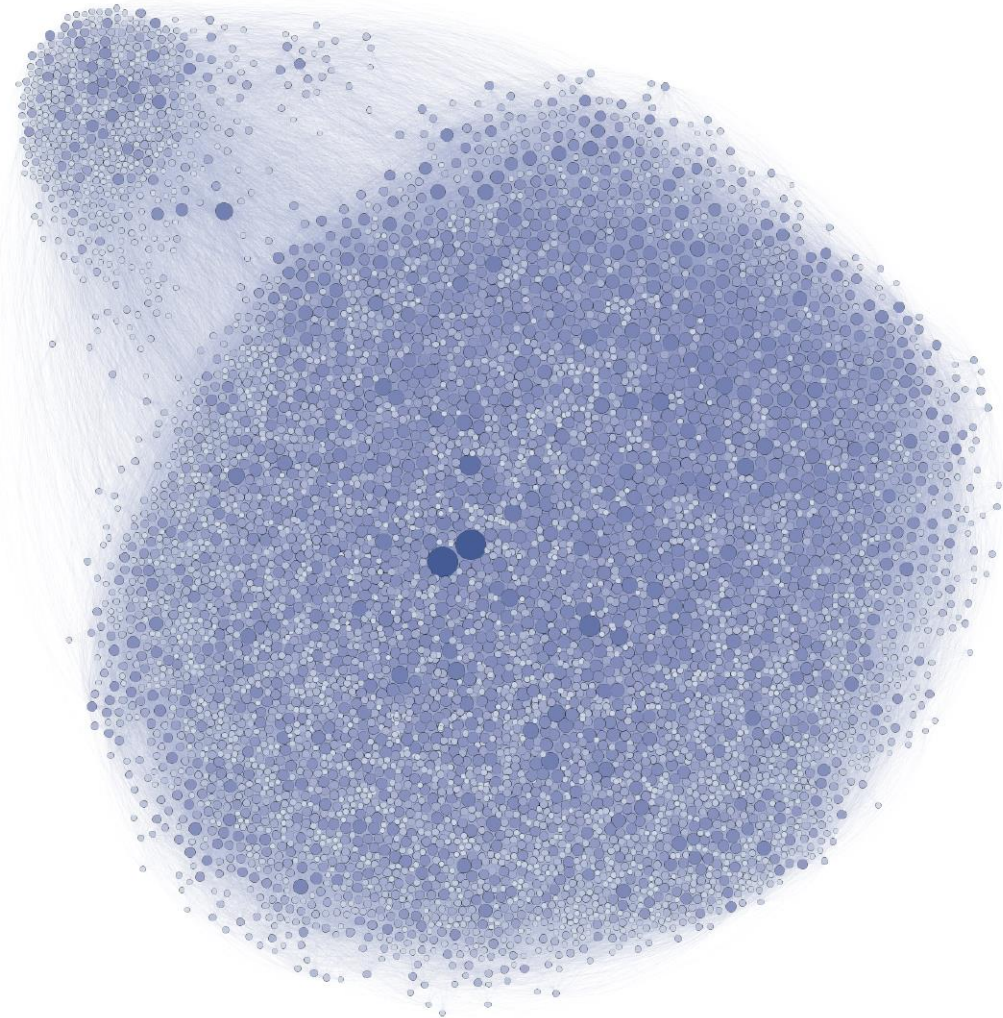
- A network of people or groups of people
 - Vertices: people or groups of people
 - Edges: connections between them, such as friendship
- Sociology has the longest and best developed tradition of empirically studying networks, and many of the tools used in network analysis today are borrowed, directly or indirectly, from sociologists.
- Today, much larger social networks can be quickly constructed using data from Facebook, Twitter (now X) or other social media platforms, enabling deeper analysis of network structure and properties in sociology
- Many interesting structural properties
 - e.g., small-world property, six degrees of separation, high clustering due to friend-to-friend triangles



Friendship network between members of a club (Newman 2010)



Social Networks - Facebook



Some people have more connections (illustrated by nodal size and color)

The network is roughly divided into two parts

Visualization of a network of Facebook users at Northwestern University. Nodes represent people, and links stand for Facebook friend connections.

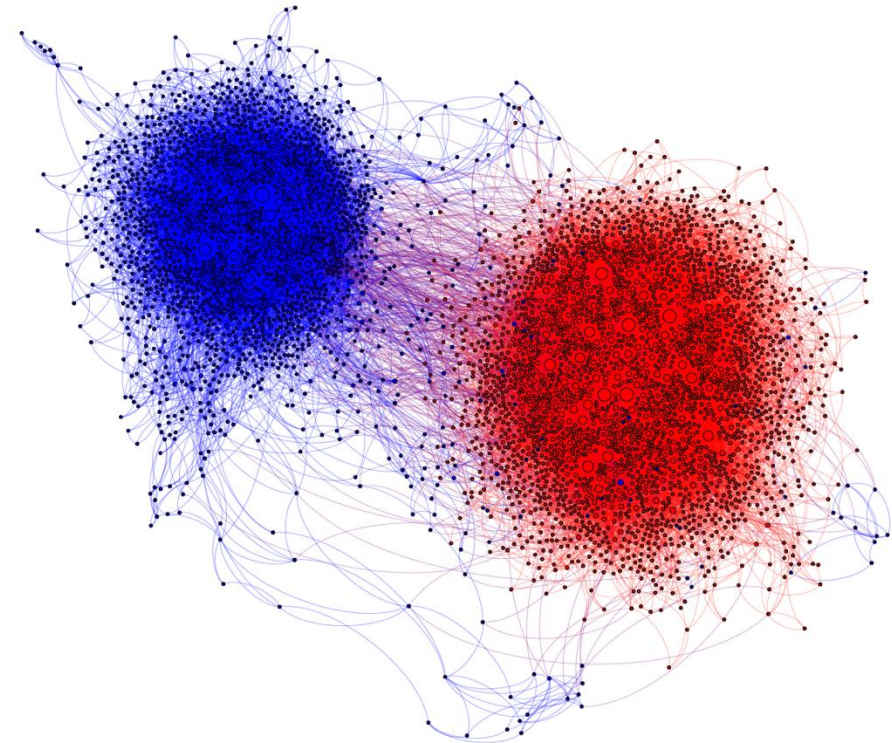
Social Networks - Twitter

- Twitter is a social network mainly aimed at spreading information – a communication (directed) network

A striking pattern: conservative users (red nodes) mostly retweet messages from other conservatives, while progressive users (blue nodes) similarly share progressive content

Network assortativity: nodes that are connected to each other in a social network tend to be similar in their features

One can thus infer a person's political preference based on the network structure



A retweet network on Twitter, among people sharing posts about US politics. Links represent retweets of posts that used hashtags such as #tcot and #p2, associated with conservative (red) and progressive (blue) messages respectively, around the 2010 US midterm election. When Bob retweets Alice, we draw a directed link from Alice to Bob to indicate that a message has propagated from her to him. The direction of the links is not shown.

Biological networks

- **Networks in the brain**

- Neural Network: neurons in the brain connected by Synapses
- Networks of macroscopic functional connectivity between large-scale regions of the brain: nodes are regions known to perform some function such as vision, motor control or learning and memory; edges denote some kind of functional connection

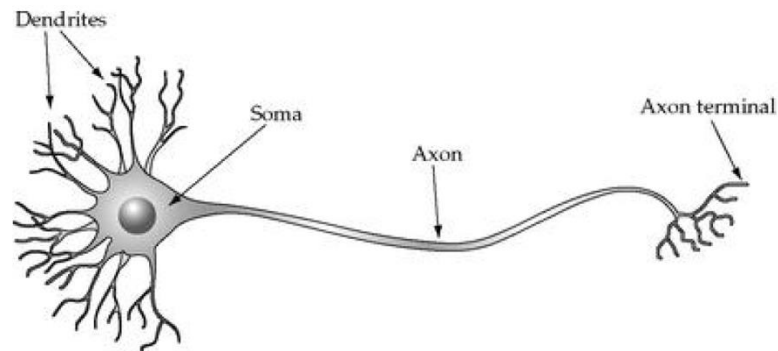


Figure 5.6: The structure of a neuron. A typical neuron is composed of a cell body or soma with many dendrites that act as inputs and a single axon that acts as an output. Towards its tip, the axon branches to allow it to connect to the inputs of several other neurons.



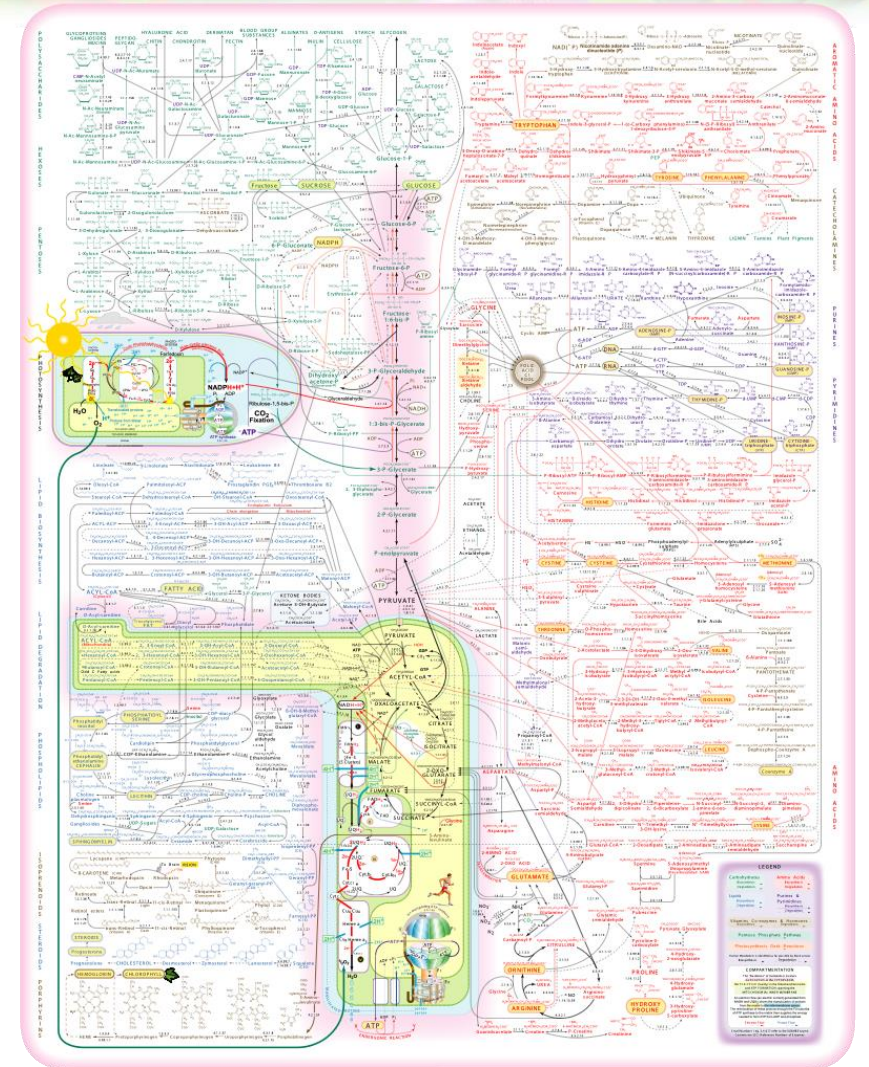
Biological networks

- **Biochemical networks** – e.g., metabolic networks, protein-protein interaction networks, gene regulatory networks
 - Vertices: proteins/genes/complexes/chemicals
 - Edges: biochemical/regulatory interactions/reactions

A metabolic network represents how metabolites (nodes) are transformed into one another via enzymatic reactions (directed edges)

At small scale/pathway level, metabolic networks can be described by Dynamical Systems on Network – modeling concentration changes over time

$$\frac{dx_i}{dt} = F_i(x_1, x_2, \dots, x_n)$$



The most correct representation of a metabolic network is as a bipartite network

Bipartite network: there are two groups of nodes such that links only connect nodes from different groups and not nodes from the same group

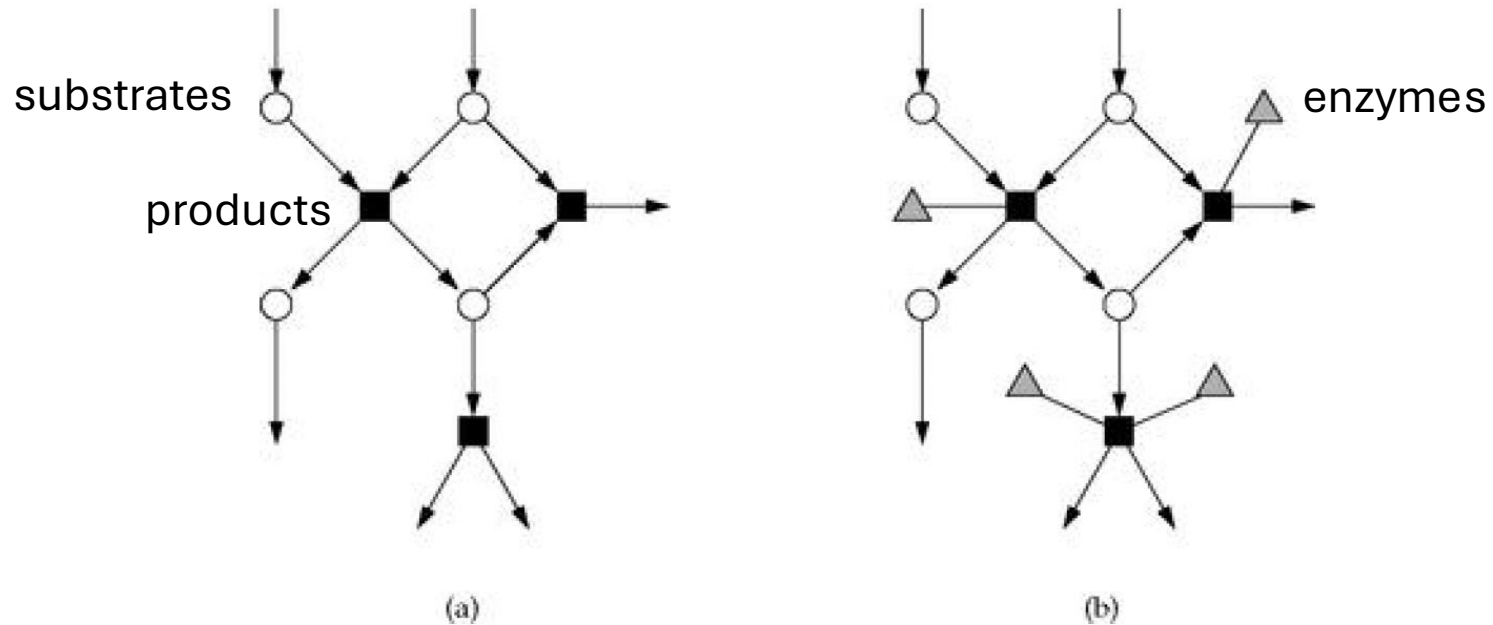


Figure 5.1: Bipartite and tripartite representations of a portion of a metabolic network.

Biological networks

- **Ecological networks** – Networks of ecological interaction between species
 - Food Webs: a directed network that represents which species prey on which others in a given ecosystem (Nodes: species; Edges: predator-prey interactions)

Almost all arrows point in one direction – Acyclic Networks

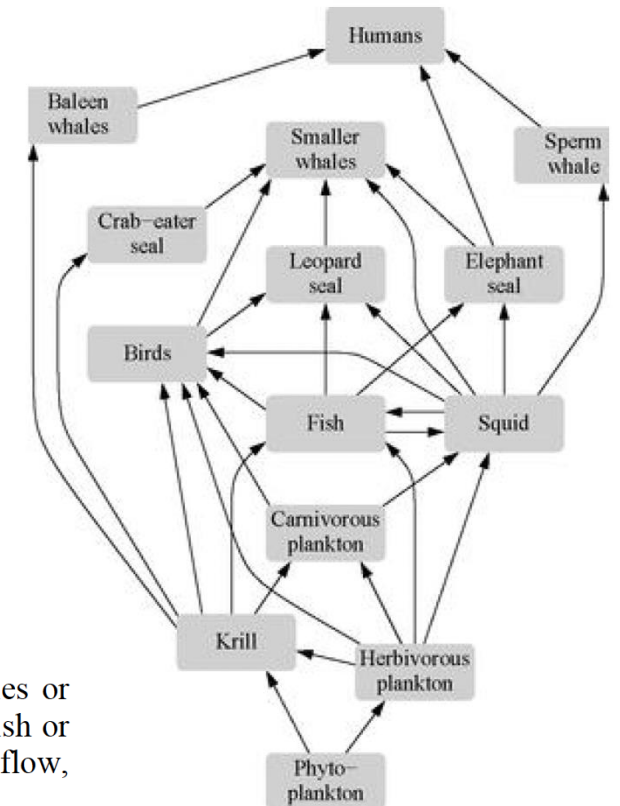
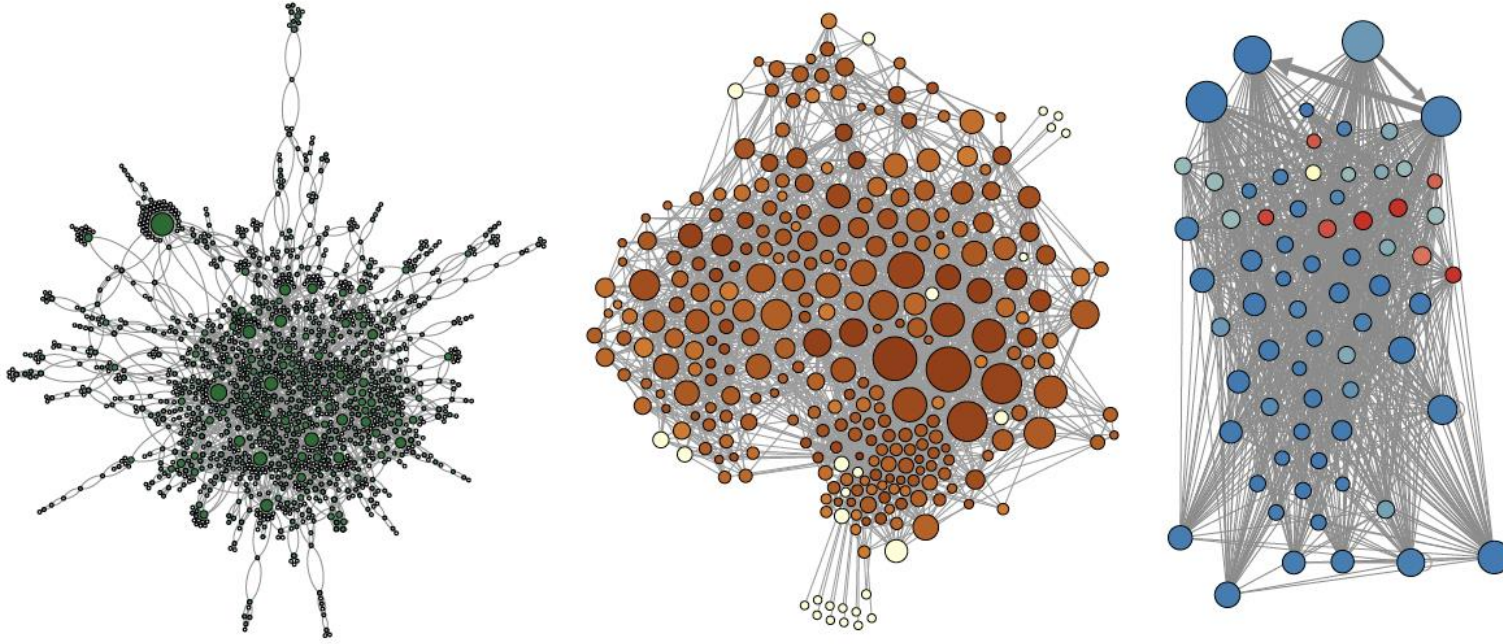


Figure 5.8: A food web of species in Antarctica. Vertices in a food web represent species or sometimes, as with some of the vertices in this diagram, groups of related species, such as fish or birds. Directed edges represent predator-prey interactions and run in the direction of energy flow, i.e., from prey to predator.

Biological networks



Three biological networks. Left: protein interaction network of yeast. Node size is proportional to the number of interacting proteins. Center: neural network of the roundworm *c. elegans*. Large and red nodes represent neurons with more outgoing and incoming synapses, respectively. Right: Food web of species in the Florida Everglades. A directed link goes from a prey to a predator species. The weight (width) of a link represents the energy flux between the two species. Node size and color represent incoming and outgoing links, respectively, so that large blue nodes are the species at the top of food chain, while small red nodes are the species at the bottom.

Dynamical Systems on Networks

Dynamical Systems on Networks

- Dynamical System: A system whose state, as represented by some set of variables, changes over time according to some given rules or equations (continuous- or discrete-time, deterministic or stochastic)
- Many real-world processes can be thought of as dynamical systems on networks:
 - Epidemics on networks (spread of infection over networks of contacts between individuals)
 - Spread of news or information between friends on social networks, movement of money through an economy
 - The flow of traffic on roads/Data over the Internet
 - Evolution of populations in an ecosystem
 - Changing concentrations of metabolites in a cell
 - Dynamics of neuronal membrane potentials in neural network
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How does nontrivial network connectivity affect network dynamics?

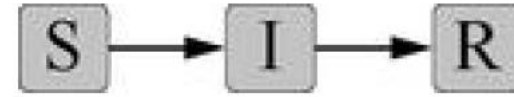
The state of each node in a network is governed by an ODE or PDE (continuous-time) or a map (discrete-time) → Studying a dynamical process on a network becomes studying system of coupled ODEs or PDEs or Maps.

One of the most important questions in network science:

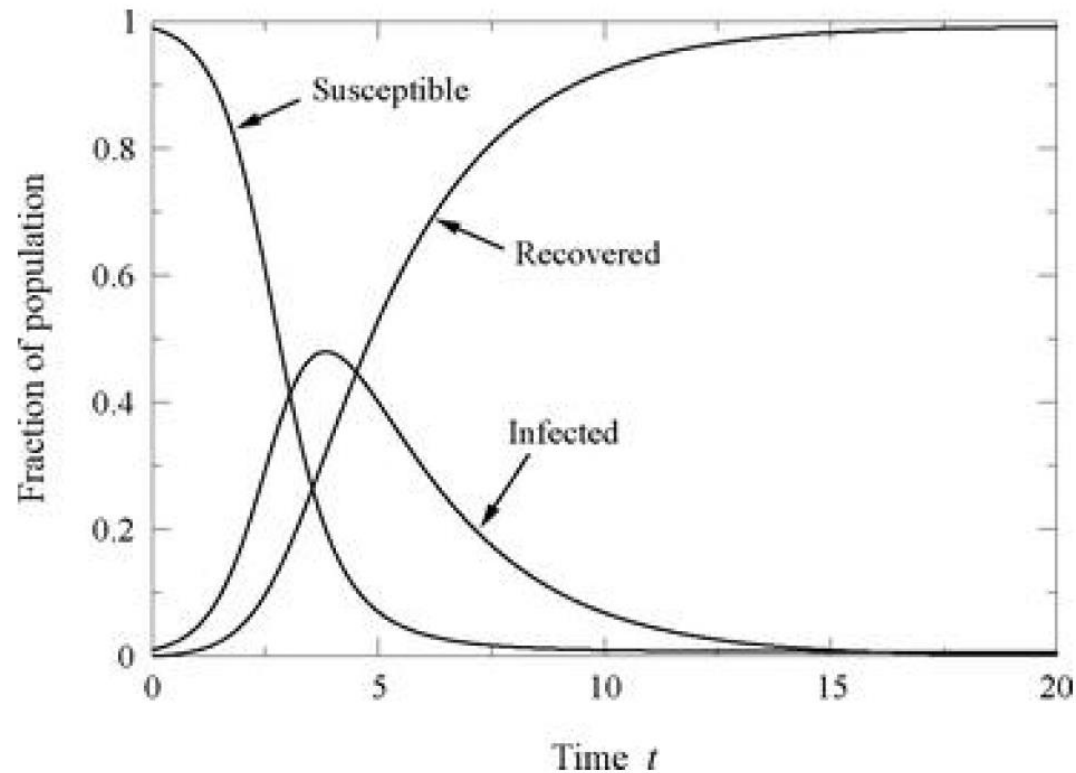
How does the underlying (nontrivial) network connectivity affect dynamical processes on networks?

- Time-independent connectivity (fixed network structure)
- Time-dependent connectivity (dynamical/adaptive networks)

Epidemic models



- Traditional Approach: SIR compartmental ODE model
- Assumes “**Fully Mixing**” of the population – each individual can potentially have contact with any other



Epidemic models on networks

- Make use of the network of potential contacts instead of assuming that contact is possible with the entire population
- Consider a stochastic Susceptible-Infected (SI) model on networks
 - Each node has two states: S or I
 - Dynamics of SI models: a stochastic process in which infection is transmitted from an infected node to a susceptible neighbor at a rate β per unit time.
 - The probability of an S node that has m infected neighbors becoming infected during the time interval dt is $1 - (1 - \beta dt)^m \rightarrow \beta m dt$ as $dt \rightarrow 0$.
 - Thus, the infection rate for an S node with m infected neighbors is βm .
 - One can also define SIS or SIR model by specifying stochastic rules for the recovery of infected nodes

Epidemic models on networks

- Though more realistic than well-mixed epidemic models, it is difficult to solve
- One important quantity to calculate (often through numerical simulations) is the total size of the disease outbreak
- In the limit of long time, the disease will spread from every initial carrier to infect all nodes in the *component* to which the carrier belongs.
- One can relate the steady-state of the basic SIR model on a network to a bond-percolation process in which one tracks occupied edges

Biological contagions → Social contagions

- Spread of ideas along social networks
- Identifying causal mechanisms of the spread of ideas is more difficult than for the spread of diseases
 - Social reinforcement
 - Homophily: agents tend to adopt the same behavior due to internal similarity, rather than because edges exist between them
 - Environment: there is a common external influence that causes agents to adopt the same behavior at different times

Coupled Oscillators

- Each node is modeled as an oscillator
- The most famous model of coupled oscillators: Kuramoto model – each node has an associated phase $\theta_i(t) \in [0, 2\pi)$:

$$\dot{\theta}_i := \frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^N b_{ij} A_{ij} f_{ij}(\theta_j - \theta_i), \quad i \in \{1, \dots, N\}.$$

- In the traditional Kuramoto model, $f_{ij}(y) = \sin(y)$ and $b_{ij} = b$ for all node pairs.
- Key question to ask:
 - How does network connectivity affect collective phenomena such as synchronization/cluster synchronization, phase-locked patterns, or chimera states where some sets of nodes synchronize with each other while other sets consist of nodes that oscillate incoherently

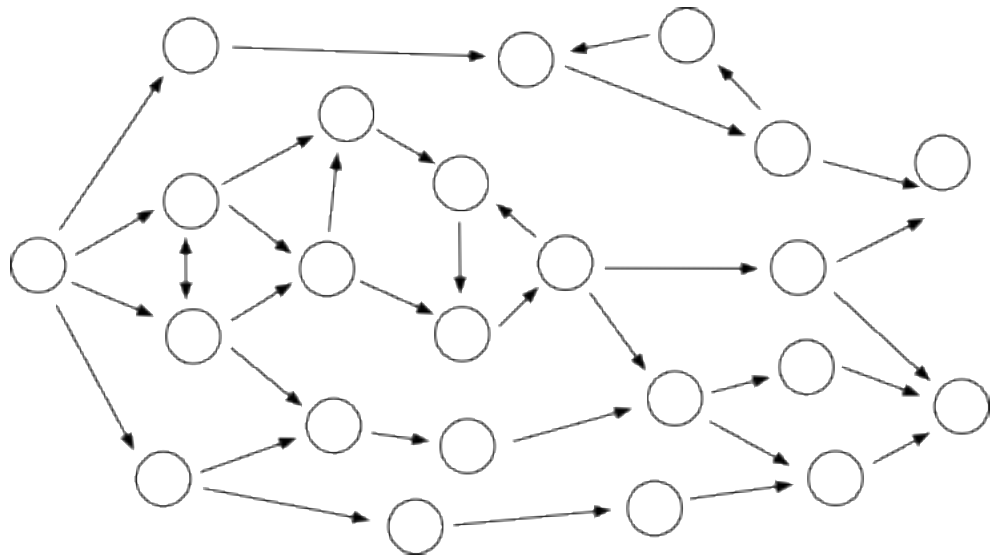
How to study dynamical systems on networks?

- Spectra of networks
 - Use eigenvalues spectra of matrices such as the adjacency matrix or the graph Laplacian to relate the qualitative behavior of a dynamical system on a network to the network structure
 - e.g., Master Stability Function
- Mean-field theories – simplify large networks into averaged dynamics
- Bifurcation theory – reduce full networks to low-dimensional networks of components associated with critical eigenvalues
- Phase reduction – simplify oscillatory dynamics to phase models for studying patterns of synchrony

How to study dynamical systems on networks?

- Coupled-cell networks – Network structure affects the dynamics that can be expected to occur generically

Coupled-Cell Networks: a directed graph



Admissible coupled ODE systems

$$\dot{x}_j = f_j(x_j, x_{i_1}, \dots, x_{i_k})$$

Coupled-cell networks: Fully inhomogeneous networks

Example1: No Hopf bifurcation in strongly feedforward network



one dimensional node — 

Admissible dynamical system:

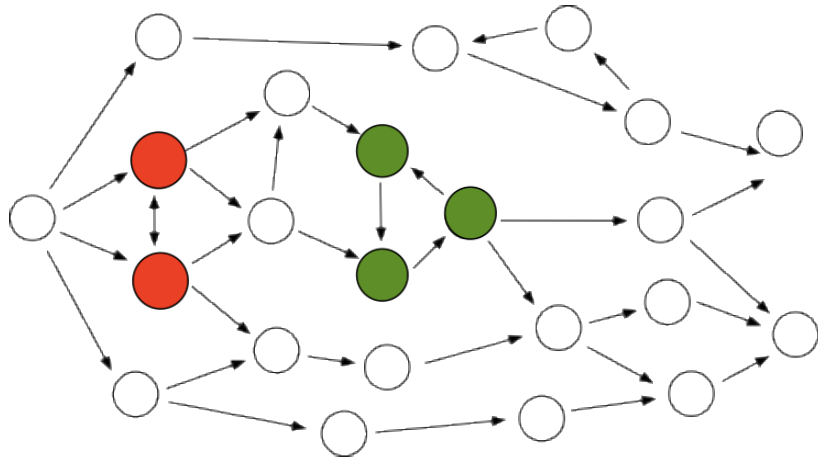
$$\begin{aligned} \dot{x}_1 &= f_1(x_1) \\ \dot{x}_2 &= f_2(x_2, x_1) \\ \dot{x}_3 &= f_3(x_3, x_2) \end{aligned}$$

Jacobian:

$$J = \begin{bmatrix} a_1 & 0 & 0 \\ b_2 & a_2 & 0 \\ 0 & b_3 & a_3 \end{bmatrix}$$

Eigenvalues of J must be real so no Hopf bifurcation.

Coupled-cell networks: fully inhomogeneous networks



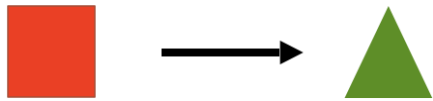
What are the generic dynamics/bifurcations in fully inhomogeneous networks?

How does network structure influence the dynamics?

Methods based on graph theory and singularity theory:

Reduced network: extract subnetwork that captures dynamics of a fully inhomogeneous network near a given bifurcation (group network into a set of path components and order them in feedforward structure)

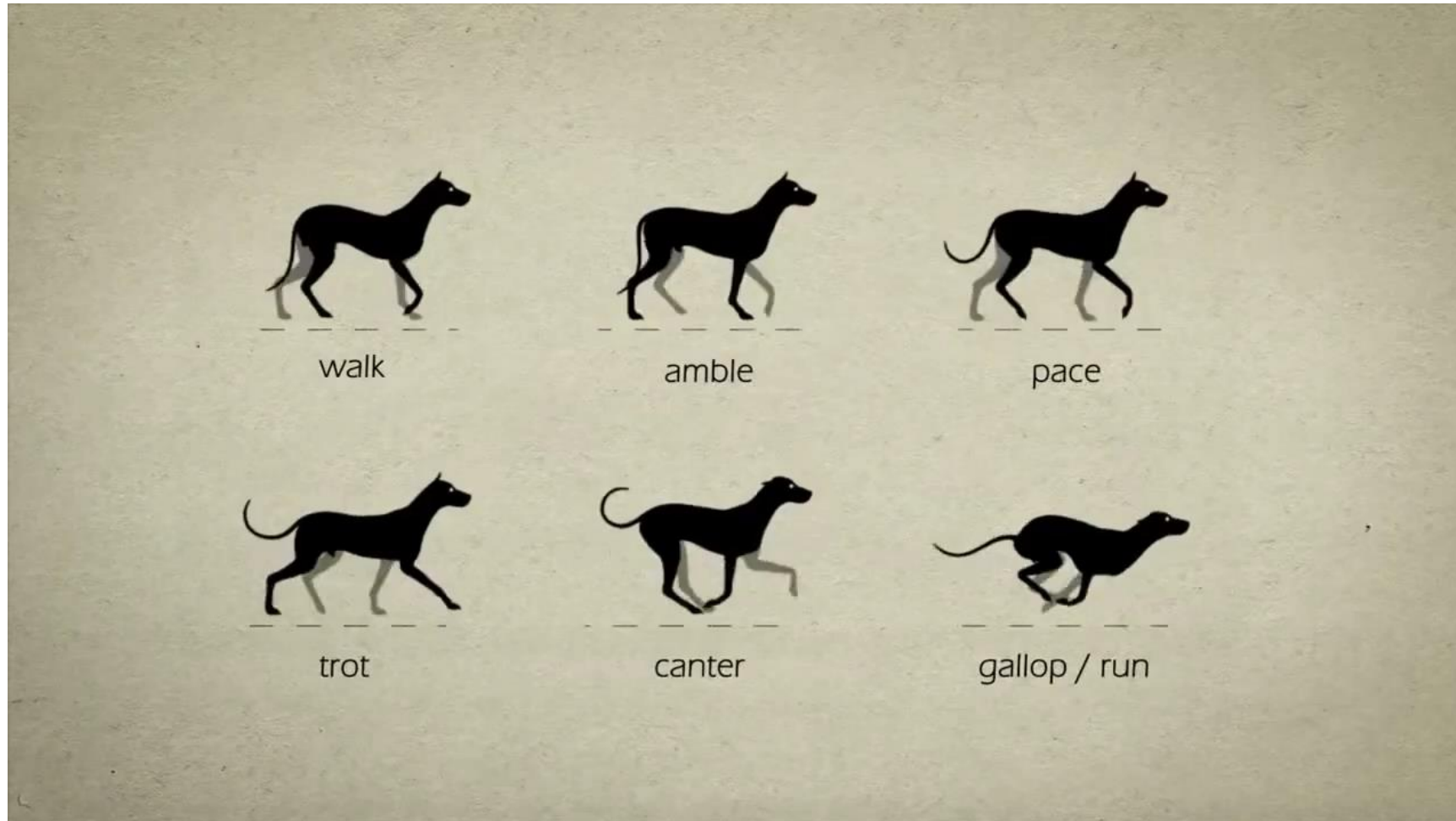
Singularity theory: analyze local bifurcations in the subnetwork



Coupled-cell networks: homogeneous networks with symmetry

- Patterns of phase-shift synchrony in periodic solutions

Quadruped Gaits



Coupled-cell networks: networks with symmetry

- Patterns of phase-shift synchrony in periodic solutions
- Quadruped gaits: predicted from network symmetry alone (H/K theory)

$$\dot{u}_i = F(u_i) + \sum_{j \rightarrow i} \alpha_{ji} H(u_j, u_i)$$

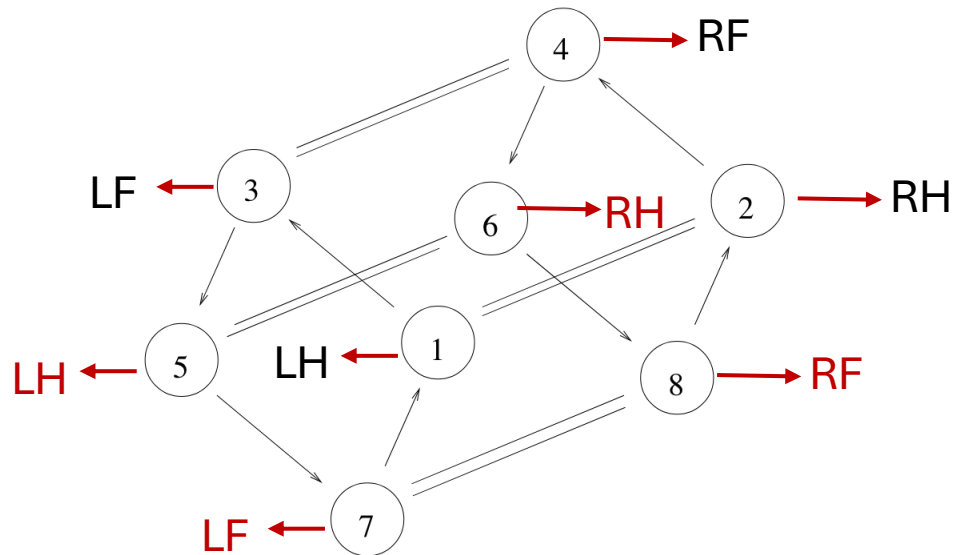


Fig. 1. Schematic eight-cell network for gaits in four-legged animals.

$Z_4 \times Z_2$ symmetry

Table 1. Phase shifts of primary gaits.

Legs	Walk		Trot		Pace	
3 4	$u_1(t + \frac{1}{4})$	$u_1(t + \frac{3}{4})$	$u_1(t + \frac{1}{2})$	$u_1(t)$	$u_1(t)$	$u_1(t + \frac{1}{2})$
1 2	$u_1(t)$	$u_1(t + \frac{1}{2})$	$u_1(t)$	$u_1(t + \frac{1}{2})$	$u_1(t)$	$u_1(t + \frac{1}{2})$
Legs	Jump		Bound		Pronk	
3 4	$u_1(t)$	$u_1(t)$	$u_1(t + \frac{1}{2})$	$u_1(t + \frac{1}{2})$	$u_1(t)$	$u_1(t)$
1 2	$u_1(t + \frac{1}{4})$	$u_1(t + \frac{1}{4})$	$u_1(t)$	$u_1(t)$	$u_1(t)$	$u_1(t)$

Models of CPGs for quadruped locomotion I. Primary gaits, Buono and Golubitsky, J. Math. Biol. 2000

Tentative plans

Part I - Core Concepts of Networks

References:

- ❖ A first course in Network Science, Menczer, Fortunato & Davis;
- ❖ Networks, Mark Newman
- **Mathematical foundations:** graphs, adjacency matrix, connectivity, path components, degree distributions, centrality measures, random graph models (Erdős–Rényi (ER), configuration model), power laws, small-world networks, etc
- **Handling Networks in Code:** use Python (NetworkX) to create, manipulate and study the structure, dynamics and function of networks - <https://github.com/CambridgeUniversityPress/FirstCourseNetworkScience>

Part II – Dynamics on Networks

References:

- ❖ Dynamical Systems on Networks, Porter & Gleeson;
- ❖ Mathematical Foundations of Neuroscience, Ermentrout & Terman;
- ❖ *Dynamics and Bifurcation in Networks: Theory and Applications of Coupled Differential Equations*, Marty Golubitsky & Ian Stewart
- **Various approaches for studying the effects of network structure on network dynamics**

- Modeling
- Master stability function
- Mean-field theory
- Phase reduction
- Formalism of coupled-cell networks
- Symmetry

References

- *A First Course in Network Science*, by Filippo Menczer, Santo Fortunato and Clayton A. Davis
- *Networks: An Introduction*, by Mark Newman
- *Dynamical Systems on Networks*, by Mason Porter & James Gleeson
- *Mathematical Foundations of Neuroscience*, Ermentrout & Terman
- *Dynamics and Bifurcation in Networks: Theory and Applications of Coupled Differential Equations*, by Marty Golubitsky & Ian Stewart